A Generalized Integral Inequality on Discontinuous Functions in the Teaching of Mathematical Analysis

Bin Zheng
Shandong University of Technology
School of Mathematics and Statistics
Zhangzhou Road 12, Zibo, 255049, China
zhengbin2601@126.com

Abstract: In this paper, we research a generalized integral inequality with two independent variables for discontinuous function in the teaching of the Mathematical Analysis course. The inequality provides an explicit bound for solutions of certain integral equations. The obtained result extends some existing results in the literature.

Key–Words: Integral inequality; discontinuous function; Integral equation; Differential equation; Bounded

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1 Introduction

Inequalities play important roles in the teaching of the college course Mathematical Analysis. In recent years many integral inequalities for continuous and discontinuous functions have been established, which provide handy tools for deriving bounds for solutions of integral and differential equations [see 1-15]. In the investigations for integral inequalities, the idea of generalizing known integral inequalities have gained extensive attention.

In [16], Pachpatte established the following integral inequality

$$(a):\ u(x,y)\le k+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}a(x,y,s,t)u(s,t)dtds+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)}b(x,y,s,t)u(s,t)dtds,$$

where $u(x,y)$ is unknown function and $u(x,y)\in C(\Delta,R_+)$, $\Delta=I_1\times I_2$, $I_1=[x_0,M]$, $I_2=[y_0,N]$, $E=\{(x,y,s,t)\in\Delta^2:\ x_0\le s\le x\le M,\ y_0\le t\le y\le N\}$, $a,\ b\in C(E,R_+)$ and $a(x,y,s,t),\ b(x,y,s,t)$ are nondecreasing in x and y for each $s\in I_1,\ t\in I_2$, $\alpha\in C^1(I_1,I_1),\ \beta\in C(I_2,I_2)$ are nondecreasing with $\alpha(x)\le x$ on I_1 and $\beta(y)\le y$ on I_2.

Recently, in [17, Theorem 3.1], the author presented the following Volterra-Fredholm type integral inequality that generalized the inequality (a).

$$(b):u(x,y)\le k+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\sigma_1(s,t)[f(s,t)\omega(u(s,t))+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t}\sigma_2(\tau,\xi)\omega(u(\tau,\xi))d\xi d\tau]dtds$$
$$+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)}\sigma_1(s,t)[f(s,t)\omega(u(s,t))+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t}\sigma_2(\tau,\xi)\omega(u(\tau,\xi))d\xi d\tau]dtds,$$

where $u(x,y)$ is unknown function and $u(x,y),\ f(x,y),\ \sigma_1(x,y),\ \sigma_2(x,y)\in C(\Delta,R_+)$, $\Delta,\ \alpha,\ \beta$ are the same as in (a).

Based on the two inequalities, some bounds for solutions of certain Volterra-Fredholm equations are derived. More details about them can be referred to read the corresponding references [16, 17].

In this paper, we will establish a new integral inequality with two independent variables for discontinuous functions which is more generalized than the inequalities mentioned above. In order to illustrate the validity of the derived result, we will present one application for it, and will derive new bounds for solutions of certain integral equations by use of the established inequality.

2 Main Results

In the rest of the paper, we denote the set of real numbers as R, and $R_+ = [0, \infty)$. $\widehat{I}$, $\widetilde{I}$ denote intervals in R, and $\widehat{I} = [x_0, A]$, $\widetilde{I} = [y_0, B]$ respectively, where $A > x_0$, $B > y_0$ are two fixed numbers.

Theorem 1 Define $\widehat{I} = [x_0, A]$, $\widetilde{I} = [y_0, B]$. Suppose $u(x, y)$ is a nonnegative continuous function on $\Omega = \bigcup_{i,j\geq 1} \Omega_{i,j}$, $\Omega_{i,j} = \{(x, y)|x_{i-1} < x \leq x_i,\ y_{j-1} < y \leq y_j\}$ with the exception in the points (x_i, y_i), $i = 1, 2, ..., n$, where there are finite jumps, and $x_0 < x_1 < ... < x_n < x_{n+1} = A$, $y_0 < y_1 < ... < y_n < y_{n+1} = B$. $\varphi(x, y)$ is a positive nondecreasing function, that is, for $\forall\ (x, y), (X, Y) \in \Omega$ and $x \leq X$, $y \leq Y$ it follows $\varphi(x, y) \leq \varphi(X, Y)$. Furthermore, suppose $q_i(x, y) \geq 1$, $i = 1, 2, 3, 4$, f_i, g_i, $h_i \in C(\widehat{I} \times \widetilde{I}, R_+)$ and $f_i(x, y) = 0$, $g_i(x, y) = 0$, $i = 1, 2$ for $(x, y) \in \Omega_{i,j}$, $i \neq j$. $\tau_1(x) \in C^1(\widehat{I}, \widehat{I})$ with $\tau_1(x) \leq x$, and $\tau_1(x) > x_i$ for $\forall x \in (x_i, x_{i+1}]$, $i = 0, 1, ..., n$. $\tau_2(y) \in C^1(\widetilde{I}, \widetilde{I})$ with $\tau_2(y) \leq y$, and $\tau(y) > y_i$ for $\forall y \in (y_i, y_{i+1}]$, $i = 0, 1, ..., n$. τ_1, τ_2 are nondecreasing.

If for $(x, y) \in \Omega_{i+1,i+1}$, $i = 0, 1, ..., n$, $u(x, y)$ satisfies the following inequality

$$\begin{aligned}
u^p(x, y) \leq\ & \varphi(x, y) + q_1(x, y) \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{\tau_1(x)} [f_1(s, t)\omega(u(s, t)) + g_1(s, t) \int_{\tau_2(y_0)}^{t} \int_{\tau_1(x_0)}^{s} h_1(\xi, \eta)\omega(u(\xi, \eta))d\xi d\eta]dsdt \\
& + q_2(x, y) \int_{\tau_2(y_0)}^{\tau_2(y_i)} \int_{\tau_1(x_0)}^{\tau_1(x)} [f_1(s, t)\omega(u(s, t)) + g_1(s, t) \int_{\tau_2(y_0)}^{t} \int_{\tau_1(x_0)}^{s} h_1(\xi, \eta)\omega(u(\xi, \eta))d\xi d\eta]dsdt \\
& + q_3(x, y) \int_{y_0}^{y} \int_{x_0}^{x} [f_2(s, t)\omega(u(s, t)) + g_2(s, t) \int_{y_0}^{t} \int_{x_0}^{s} h_2(\xi, \eta)\omega(u(\xi, \eta))d\xi d\eta]dsdt \\
& + q_4(x, y) \int_{y_0}^{y_i} \int_{x_0}^{x} [f_2(s, t)\omega(u(s, t)) + g_2(s, t) \int_{y_0}^{t} \int_{x_0}^{s} h_2(\xi, \eta)\omega(u(\xi, \eta))d\xi d\eta]dsdt \\
& + \sum_{x_0 < x_j < x, y_0 < y_j < y} \beta_j u(x_j - 0, y_j - 0), \qquad (1)
\end{aligned}$$

then

$$\begin{aligned}
u(x, y) \leq\ & \{\widetilde{G}_i^{-1}\{\widetilde{G}_i(N_i(x, y_{i+1})) + \int_{\tau_2(y_i)}^{\tau_2(y)} \int_{\tau_1(x_i)}^{\tau_1(x)} \frac{f_1(s, t)}{\varphi(s, t)} \omega[(\varphi(s, t)q(s, t))^{\frac{1}{p}}]dsdt \\
& + \int_{\tau_2(y_i)}^{\tau_2(y)} \int_{\tau_1(x_i)}^{\tau_1(x)} g_1(s, t) \int_{\tau_2(y_0)}^{t} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi, \eta)}{\varphi(\xi, \eta)} \omega[(\varphi(\xi, \eta)q(\xi, \eta))^{\frac{1}{p}}]d\xi d\eta dsdt \\
& + \int_{y_i}^{y} \int_{x_i}^{x} [\frac{f_2(s, t)}{\varphi(s, t)} \omega[(\varphi(s, t)q(s, t))^{\frac{1}{p}}] + g_2(s, t) \int_{y_0}^{t} \int_{x_0}^{s} \frac{h_2(\xi, \eta)}{\varphi(\xi, \eta)} \omega[(\varphi(\xi, \eta)q(\xi, \eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}\varphi(x, y)q(x, y)\}^{\frac{1}{p}}
\end{aligned}$$

holds for $(x, y) \in \Omega_{i+1,i+1}$, $i = 0, 1..., n$, (2)

where

$$\widetilde{G}_i(v) = \int_{l_i}^{v} \frac{1}{\omega(s^{\frac{1}{p}}(t))} ds,\ i = 0, 1, ..., n, \qquad (3)$$

$$\widetilde{H}_i(t) = \widetilde{G}_i(2t - l_i) - \widetilde{G}_i(t), \qquad (4)$$

and H_i are nondecreasing on $t \geq l_i$, $i = 0, 1, ..., n$,

$$l_0 = 1,\ l_i = \gamma_i + \beta_i[\varphi(x_i - 0, y_i - 0)q(x_i - 0, y_i - 0)\gamma_i]^{\frac{1}{p}},\ i = 1, 2, ..., n, \qquad (5)$$

$$\begin{aligned}
\gamma_i =\ & \widetilde{G}_{i-1}^{-1}\{\widetilde{G}_{i-1}(N_{i-1}(x_i, y_i)) + \int_{\tau_2(y_{i-1})}^{\tau_2(y_i)} \int_{\tau_1(x_{i-1})}^{\tau_1(x_i)} \frac{f_1(s, t)}{\varphi(s, t)} \omega[(\varphi(s, t)q(s, t))^{\frac{1}{p}}]dsdt \\
& + \int_{\tau_2(y_{i-1})}^{\tau_2(y_i)} \int_{\tau_1(x_{i-1})}^{\tau_1(x_i)} g_1(s, t) \int_{\tau_2(y_0)}^{t} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi, \eta)}{\varphi(\xi, \eta)} \omega[(\varphi(\xi, \eta)q(\xi, \eta))^{\frac{1}{p}}]d\xi d\eta dsdt \\
& + \int_{y_{i-1}}^{y_i} \int_{x_{i-1}}^{x_i} [\frac{f_2(s, t)}{\varphi(s, t)} \omega[(\varphi(s, t)q(s, t))^{\frac{1}{p}}] + g_2(s, t) \int_{y_0}^{t} \int_{x_0}^{s} \frac{h_2(\xi, \eta)}{\varphi(\xi, \eta)} \omega[(\varphi(\xi, \eta)q(\xi, \eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\},
\end{aligned}$$

$i = 1, 2, ..., n,$ (6)

$$N_{i-1}(x, y) = \widetilde{H}_{i-1}^{-1}\{\int_{\tau_2(y_{i-1})}^{\tau_2(y)} \int_{\tau_1(x_{i-1})}^{\tau_1(x)} \frac{f_1(s, t)}{\varphi(s, t)} \omega[(\varphi(s, t)q(s, t))^{\frac{1}{p}}]dsdt$$

$$+\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds dt$$
$$+\int_{y_{i-1}}^{y}\int_{x_{i-1}}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]ds dt\},$$
$$i=1,2,...,n+1. \quad (7)$$

Proof: Define

$$\widetilde{v}_i(x,y)=l_i+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{\tau_2(y_i)}^{\tau_2(y_{i+1})}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{y_i}^{y}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{y_i}^{y_{i+1}}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt,\ i=0,1,...,n.$$

Let $q(x,y)=max\{q_i(x,y),\ i=1,2,3,4\}$ and $v(x,y)=\dfrac{u^p(x,y)}{\varphi(x,y)q(x,y)}$, then obviously $q(x,y)\geq 1$. Considering $\varphi(x,y)$ is nondecreasing, then from (1) we can easily deduce

$$v(x,y)\leq 1+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{y_0}^{y_1}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\sum_{x_0<x_i<x,y_0<y_i<y}\beta_i\psi(\phi(u(x_i-0,y_i-0))). \quad (8)$$

Case 1: If $(x,y)\in\Omega_{11}$, considering $l_0=1$, then from (8) it follows

$$v(x,y)\leq l_0+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt$$
$$+\int_{y_0}^{y_1}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds dt=\widetilde{v}_0(x,y). \quad (9)$$

Given a fixed $X\in(x_0,x_1]$ and $x\in(x_0,X]$, then

$$v(x,y)\leq\widetilde{v}_0(x,y)\leq\widetilde{v}_0(X,y), \quad (10)$$

and

$$(\widetilde{v}_0(X,y))'_y=\tau'_2(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)}[\frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega(u(s,\tau_2(y)))+g_1(s,\tau_2(y))\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds\}$$
$$+\int_{x_0}^{X}[\frac{f_2(s,y)}{\varphi(s,y)}\omega(u(s,y))+g_2(s,y)\int_{y_0}^{y}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]ds$$
$$=\tau'_2(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)}\frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega[(\varphi(s,\tau_2(y))q(s,\tau_2(y)))^{\frac{1}{p}}(v(s,\tau_2(y)))^{\frac{1}{p}}]ds\}$$
$$+\tau'_2(y)\{\int_{x_0}^{\tau_1(X)}g_1(s,\tau_2(y))\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}(v(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}$$
$$+\int_{x_0}^{X}\frac{f_2(s,y)}{\varphi(s,y)}\omega[(\varphi(s,y)q(s,y))^{\frac{1}{p}}(v(s,y))^{\frac{1}{p}}]ds$$
$$+\int_{x_0}^{X}g_2(s,y)\int_{y_0}^{y}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}(v(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds$$
$$\leq\tau'_2(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)}\frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega[(\varphi(s,\tau_2(y))q(s,\tau_2(y)))^{\frac{1}{p}}]\omega[((v(s,\tau_2(y)))^{\frac{1}{p}}]ds\}$$

$$
\begin{aligned}
&+\tau_2'(y)\{\int_{x_0}^{\tau_1(X)} g_1(s,\tau_2(y)) \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]\omega[(v(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}\\
&+\int_{x_0}^{X} \frac{f_2(s,y)}{\varphi(s,y)}\omega[(\varphi(s,y)q(s,y))^{\frac{1}{p}}]\omega[(v(s,y))^{\frac{1}{p}}]ds\\
&+\int_{x_0}^{X} g_2(s,y) \int_{y_0}^{y} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]\omega[(v(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\\
\leq\ &\tau_2'(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)} \frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega[(\varphi(s,\tau_2(y))q(s,\tau_2(y)))^{\frac{1}{p}}]\omega[((\widetilde{v}_0(s,\tau_2(y)))^{\frac{1}{p}}]ds\}\\
&+\tau_2'(y)\{\int_{x_0}^{\tau_1(X)} g_1(s,\tau_2(y)) \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]\omega[(\widetilde{v}_0(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}\\
&+\int_{x_0}^{X} \frac{f_2(s,y)}{\varphi(s,y)}\omega[(\varphi(s,y)q(s,y))^{\frac{1}{p}}]\omega[(\widetilde{v}_0(s,y))^{\frac{1}{p}}]ds\\
&+\int_{x_0}^{X} g_2(s,y) \int_{y_0}^{y} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]\omega[(\widetilde{v}_0(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\\
\leq\ &\tau_2'(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)} \frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega[(\varphi(s,\tau_2(y))q(s,\tau_2(y)))^{\frac{1}{p}}]ds\}\omega[((\widetilde{v}_0(\tau_1(X),\tau_2(y)))^{\frac{1}{p}}]\\
&+\tau_2'(y)\{\int_{x_0}^{\tau_1(X)} g_1(s,\tau_2(y)) \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}\omega[(\widetilde{v}_0(\tau_1(X),\tau_2(y)))^{\frac{1}{p}}]\\
&+\{\int_{x_0}^{X} \frac{f_2(s,y)}{\varphi(s,y)}\omega[(\varphi(s,y)q(s,y))^{\frac{1}{p}}]ds\}\omega[(\widetilde{v}_0(X,y))^{\frac{1}{p}}]\\
&+\{\int_{x_0}^{X} g_2(s,y) \int_{y_0}^{y} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}\omega[(\widetilde{v}_0(X,y))^{\frac{1}{p}}]\\
\leq\ &\tau_2'(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)} \frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega[(\varphi(s,\tau_2(y))q(s,\tau_2(y)))^{\frac{1}{p}}]ds\}\omega[((\widetilde{v}_0(X,y))^{\frac{1}{p}}]\\
&+\tau_2'(y)\{\int_{x_0}^{\tau_1(X)} g_1(s,\tau_2(y)) \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}\omega[(\widetilde{v}_0(X,y))^{\frac{1}{p}}]\\
&+\{\int_{x_0}^{X} \frac{f_2(s,y)}{\varphi(s,y)}\omega[(\varphi(s,y)q(s,y))^{\frac{1}{p}}]ds\}\omega[(\widetilde{v}_0(X,y))^{\frac{1}{p}}]\\
&+\{\int_{x_0}^{X} g_2(s,y) \int_{y_0}^{y} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}\omega[(\widetilde{v}_0(X,y))^{\frac{1}{p}}], \qquad (11)
\end{aligned}
$$

that is,

$$
\begin{aligned}
\frac{(\widetilde{v}_0(X,y))_y'}{\omega[(\widetilde{v}_0(X,y))^{\frac{1}{p}}]} \leq\ &\tau_2'(y)\{\int_{\tau_1(x_0)}^{\tau_1(X)} \frac{f_1(s,\tau_2(y))}{\varphi(s,\tau_2(y))}\omega[(\varphi(s,\tau_2(y))q(s,\tau_2(y)))^{\frac{1}{p}}]ds\}\\
&+\tau_2'(y)\{\int_{x_0}^{\tau_1(X)} g_1(s,\tau_2(y)) \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta ds\}]\\
&+\int_{x_0}^{X}\{\frac{f_2(s,y)}{\varphi(s,y)}\omega[(\varphi(s,y)q(s,y))^{\frac{1}{p}}]+g_2(s,y) \int_{y_0}^{y} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta\}ds. \qquad (12)
\end{aligned}
$$

An integration for (12) with respect to y from y_0 to y yields

$$
\begin{aligned}
&\widetilde{G}_0[\widetilde{v}_0(X,y)] - \widetilde{G}_0[\widetilde{v}_0(X,y_0)]\\
&\leq \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{\tau_1(X)}\{\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_1(s,t) \int_{\tau_2(y_0)}^{t} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta\}dsdt\\
&\quad+\int_{y_0}^{y} \int_{x_0}^{X}\{\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}] + g_2(s,t) \int_{y_0}^{t} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta\}dsdt,
\end{aligned}
$$

Since G_0 is nondecreasing, it follows

$$
\begin{aligned}
\widetilde{v}_0(X,y) \leq\ &\widetilde{G}_0^{-1}\{\widetilde{G}_0[\widetilde{v}_0(X,y_0)] + \int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{\tau_1(X)} \frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt\\
&+\int_{\tau_2(y_0)}^{\tau_2(y)} \int_{\tau_1(x_0)}^{\tau_1(X)} g_1(s,t) \int_{\tau_2(y_0)}^{t} \int_{\tau_1(x_0)}^{s} \frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt\\
&+\int_{y_0}^{y} \int_{x_0}^{X}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t) \int_{y_0}^{t} \int_{x_0}^{s} \frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \qquad (13)
\end{aligned}
$$

Considering (13) holds for $\forall y \in (y_0, y_1]$, then combining (9) and (13) we have

$$
2\widetilde{v}_0(X,y_0) - l_0 = \widetilde{v}_0(X,y_1) \leq \widetilde{G}_0^{-1}\{\widetilde{G}_0[\widetilde{v}_0(X,y_0)] + \int_{\tau_2(y_0)}^{\tau_2(y_1)} \int_{\tau_1(x_0)}^{\tau_1(X)} \frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt
$$

$$+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(X)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y_1}\int_{x_0}^{X}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}, \quad (14)$$

that is,

$$\widetilde{G}_0[2\widetilde{v}_0(X,y_0)-l_0]-\widetilde{G}_0[\widetilde{v}_0(X,y_0)]\le\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(X)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(X)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y_1}\int_{x_0}^{X}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt, \quad (15)$$

Since $\widetilde{H}_0(t)=\widetilde{G}_0(2t-l_0)-\widetilde{G}_0(t)$, and $\widetilde{H}_0(t)$ is nondecreasing on $t\ge l_0$, considering $\widetilde{v}_0(X,y_0)\ge 1$, then we have

$$\widetilde{v}_0(X,y_0)\le\widetilde{H}_0^{-1}\{\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(X)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(X)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y_1}\int_{x_0}^{X}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}$$
$$=N_0(X,y_1). \quad (16)$$

Combining (10), (13) and (16) we obtain

$$v(x,y)\le\widetilde{v}_0(X,y)\le\widetilde{G}_0^{-1}\{\widetilde{G}_0(N_0(X,y_1))+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(X)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(X)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \quad (17)$$

Since (17) holds for $x\in(x_0,X]$, setting $x=X$ in (17), it follows

$$v(X,y)\le\widetilde{v}_0(X,y)\le\widetilde{G}_0^{-1}\{\widetilde{G}_0(N_0(X,y_1))+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(X)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(X)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \quad (18)$$

Considering X is selected from $(x_0,x_1]$ arbitrarily, substituting X with x, and we obtain

$$v(x,y)\le\widetilde{v}_0(x,y)\le\widetilde{G}_0^{-1}\{\widetilde{G}_0(N_0(x,y_1))+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \quad (19)$$

So

$$\widetilde{v}_0(x_1,y_1)\le\widetilde{G}_0^{-1}\{\widetilde{G}_0(N_0(x_1,y_1))+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(x_1)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(x_1)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_0}^{y_1}\int_{x_0}^{x_1}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}=$$

γ_1, and then

$$u(x_1-0,y_1-0)=[\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)v(x_1-0,y_1-0)]^{\frac{1}{p}}$$

$$\leq [\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)\widetilde{v}_0(x_1-0,y_1-0)]^{\frac{1}{p}}$$
$$\leq [\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)\widetilde{v}_0(x_1,y_1)]^{\frac{1}{p}}$$
$$\leq [\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)\gamma_1]^{\frac{1}{p}}.$$

Case 2: If $(x,y)\in\Omega_{22}$, then from (8) it follows

$$\begin{aligned}
v(x,y) &\leq 1+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{\tau_2(y_0)}^{\tau_2(y_2)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_0}^{y}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_0}^{y_2}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt+\beta_1 u(x_1-0,y_1-0)\\
&=1+2\int_{\tau_2(y_0)}^{\tau_2(y_1)}\int_{\tau_1(x_0)}^{\tau_1(x_1)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+2\int_{y_0}^{y_1}\int_{x_0}^{x_1}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&\int_{\tau_2(y_1)}^{\tau_2(y)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{\tau_2(y_1)}^{\tau_2(y_2)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y_2}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt+\beta_1 u(x_1-0,y_1-0)\\
&=\widetilde{v}_0(x_1,y_1)+\int_{\tau_2(y_1)}^{\tau_2(y)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{\tau_2(y_1)}^{\tau_2(y_2)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y_2}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt+\beta_1 u(x_1-0,y_1-0)\\
&\leq \gamma_1+\beta_1[\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)\gamma_1]^{\frac{1}{p}}\\
&+\int_{\tau_2(y_1)}^{\tau_2(y)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{\tau_2(y_1)}^{\tau_2(y_2)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y_2}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&=l_1+\int_{\tau_2(y_1)}^{\tau_2(y)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{\tau_2(y_1)}^{\tau_2(y_2)}\int_{\tau_1(x_1)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt\\
&+\int_{y_1}^{y_2}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt=\widetilde{v}_1(x,y).
\end{aligned}\tag{20}$$

We notice (20) is similar to (9) in structure. Then following in the same process as (10)-(19) we obtain

$$\begin{aligned}
v(x,y) &\leq \widetilde{v}_1(x,y)\leq \widetilde{G}_1^{-1}\{\widetilde{G}_1(N_1(x,y_2))+\int_{\tau_2(y_1)}^{\tau_2(y)}\int_{\tau_1(x_1)}^{\tau_1(x)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt\\
&+\int_{\tau_2(y_1)}^{\tau_2(y)}\int_{\tau_1(x_1)}^{\tau_1(x)}g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt
\end{aligned}$$

$$+\int_{y_1}^{y}\int_{x_1}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \quad (21)$$

Case 3: If $(x,y)\in\Omega_{ii}$, the following inequality holds

$$v(x,y)\le\widetilde{v}_{i-1}(x,y)\le\widetilde{G}_{i-1}^{-1}\{\widetilde{G}_{i-1}(N_{i-1}(x,y_i))+\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)}g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_{i-1}}^{y}\int_{x_{i-1}}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \quad (22)$$

Then for $(x,y)\in\Omega_{i+1,i+1}$, from (8) it follows

$$v(x,y)\le 1+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_{i+1})}\int_{\tau_1(x_0)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_0}^{y_{i+1}}\int_{x_0}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j u(x_j-0,y_j-0)$$
$$=1+2\int_{\tau_2(y_0)}^{\tau_2(y_i)}\int_{\tau_1(x_0)}^{\tau_1(x_i)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+2\int_{y_0}^{y_i}\int_{x_0}^{x_i}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{\tau_2(y_i)}^{\tau_2(y_{i+1})}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_i}^{y}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_i}^{y_{i+1}}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j u(x_j-0,y_j-0)$$
$$\le\gamma_i+\beta_i[\varphi(x_i-0,y_i-0)q(x_i-0,y_i-0)\gamma_i]^{\frac{1}{p}}$$
$$+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{\tau_2(y_i)}^{\tau_2(y_{i+1})}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_i}^{y}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_i}^{y_{i+1}}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$=l_i+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{\tau_2(y_i)}^{\tau_2(y_{i+1})}\int_{\tau_1(x_i)}^{\tau_1(x)}[\frac{f_1(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_i}^{y}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\int_{y_i}^{y_{i+1}}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega(u(s,t))+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega(u(\xi,\eta))d\xi d\eta]dsdt=\widetilde{v}_i(x,y). \quad (23)$$

Then similar to (10)-(19) we obtain

$$v(x,y)\le\widetilde{v}_i(x,y)\le\widetilde{G}_i^{-1}\{\widetilde{G}_i(N_i(x,y_{i+1}))+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}\frac{f_1(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$

$$+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)} g_1(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h_1(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt$$
$$+\int_{y_i}^{y}\int_{x_i}^{x}[\frac{f_2(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]+g_2(s,t)\int_{y_0}^{t}\int_{x_0}^{s}\frac{h_2(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta]dsdt\}. \quad (24)$$

Considering $u(x,y)=(v(x,y)\varphi(x,y)q(x,y))^{\frac{1}{p}}$, then the proof is complete.

Corollary 2 Suppose $u,\ \varphi,\ \omega,\ \tau_1,\ \tau_2,\ q_i,\ i=1,2,\ \beta_j,\ j=1,2,...,n,$ and $f,\ g,\ h\in C(\widehat{I}\times\widetilde{I})$. If for $(x,y)\in\Omega_{i+1,i+1},\ i=0,1,...,n,\ u(x,y)$ satisfies the following inequality

$$u^p(x,y)\le\varphi(x,y)+q_1(x,y)\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[f(s,t)\omega(u(s,t))+g(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}h(\xi,\eta)\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+q_2(x,y)\int_{\tau_2(y_0)}^{\tau_2(y_i)}\int_{\tau_1(x_0)}^{\tau_1(x)}[f(s,t)\omega(u(s,t))+g(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}h(\xi,\eta)\omega(u(\xi,\eta))d\xi d\eta]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j u(x_j-0,y_j-0), \quad (25)$$

then

$$u(x,y)\le\{\widetilde{G}_i^{-1}\{\widetilde{G}_i(N_i(x,y_{i+1}))+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}\frac{f(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}g(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt\}\varphi(x,y)q(x,y)\}^{\frac{1}{p}}$$

holds for $(x,y)\in\Omega_{i+1,i+1},\ i=0,1...,n,$ (26)
where

$$\widetilde{G}_i(v)=\int_{l_i}^{v}\frac{1}{\omega(s^{\frac{1}{p}}(t))}ds,\ i=0,1,...,n,$$
$\widetilde{H}_i(t)=\widetilde{G}_i(2t-l_i)-\widetilde{G}_i(t)$, and H_i are nondecreasing on $t\ge l_i,\ i=0,1,...,n,$
$$l_0=1,\ l_i=\gamma_i+\beta_i[\varphi(x_i-0,y_i-0)q(x_i-0,y_i-0)\gamma_i]^{\frac{1}{p}},\ i=1,2,...,n,$$
$$\gamma_i=\widetilde{G}_{i-1}^{-1}\{\widetilde{G}_{i-1}(N_{i-1}(x_i,y_i))+\int_{\tau_2(y_{i-1})}^{\tau_2(y_i)}\int_{\tau_1(x_{i-1})}^{\tau_1(x_i)}\frac{f(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_{i-1})}^{\tau_2(y_i)}\int_{\tau_1(x_{i-1})}^{\tau_1(x_i)}g(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt\},\quad i=1,2,...,n,$$
$$N_{i-1}(x,y)=\widetilde{H}_{i-1}^{-1}\{\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)}\frac{f(s,t)}{\varphi(s,t)}\omega[(\varphi(s,t)q(s,t))^{\frac{1}{p}}]dsdt$$
$$+\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)}g(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\frac{h(\xi,\eta)}{\varphi(\xi,\eta)}\omega[(\varphi(\xi,\eta)q(\xi,\eta))^{\frac{1}{p}}]d\xi d\eta dsdt\},\quad i=1,2,...,n+1.$$

If we take $\tau_1(x)=x,\ \tau_2(y)=y$ in Corollary 2 then we can obtain another corollary, which is left to the readers.

Remark If we take $\varphi(x,y)\equiv C>0$, and $u(x,y)$ is continuous on Ω, then Theorem 1 reduces to [17, Theorem 3.1] with slight difference. Furthermore, if we take $\omega(u)=u$, then Corollary 2 reduces Pachpatte's result in [16].

3 Application

In this section, we will use the established inequality above to derive explicit bounds for certain integral equation.

Consider the retarded Volterra-Fredholm integral equation with two independent variables of the form

$$u^p(x,y)=\varphi(x,y)+\int_{y_0}^{y}\int_{x_0}^{x}M_1[s,t,u(\tau_1(s),\tau_2(t)),\int_{y_0}^{t}\int_{x_0}^{s}N_1(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta]dsdt$$
$$+\int_{y_0}^{y_i}\int_{x_0}^{x}M_2[s,t,u(\tau_1(s),\tau_2(t)),\int_{y_0}^{t}\int_{x_0}^{s}N_2(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j u(x_j-0,y_j-0),\quad\forall(x,y)\in\Omega_{i,i},\ i=1,2,...,n+1 \quad (27)$$

where $u(x,y)$ is a continuous function on $\Omega = \bigcup\limits_{i,j\geq 1} \Omega_{i,j},\ \Omega_{i,j} = \{(x,y)|x_{i-1} < x \leq x_i,\ y_{j-1} < y \leq y_j\}$ with the exception in the points $(x_i, y_i),\ i = 1,2,...,n$, where there are finite jumps, and $x_0 < x_1 < ... < x_n < x_{n+1} = A,\ y_0 < y_1 < ... < y_n < y_{n+1} = B$. $\tau_1(x) \in C^1(I,I)$ with $\tau_1(x) \leq x$, and $\tau_1(x) > x_i$ for $\forall x \in (x_i, x_{i+1}],\ i = 0,1,...,n$. $\tau_2(y) \in C^1(\widetilde{I},\widetilde{I})$ with $\tau_2(y) \leq y$, and $\tau(y) > y_i$ for $\forall y \in (y_i, y_{i+1}],\ i = 0,1,...,n$. $\tau_1,\ \tau_2$ are strictly increasing.

Theorem 3: Assume $u(x,y)$ is a solution of (27), and the following conditions satisfies

$$\begin{cases} |\varphi(x,y)| \leq C \\ |M_i(s,t,x,y)| \leq f(s,t)|x|^q + g(s,t)|y|,\ i = 1,2 \\ |N_i(s,t,x)| \leq h(s,t)|x|^q,\ i = 1,2 \\ \omega(v) = v^q \end{cases} \tag{28}$$

where $f(x,y),\ g(x,y),\ h(x,y) \in C(\widehat{I} \times \widetilde{I}, R_+)$ and $f(x,y) = 0,\ g(x,y) = 0$ for $(x,y) \in \Omega_{i,j},\ i \neq j$, $\omega \in C(R_+, R_+)$, q is a constant with $0 < q < p$, and $C > 0$ is a constant, then

$$u(x,y) \leq [\varphi(x,y)q(x,y)]^{\frac{1}{p}}\{\frac{p-q}{p}\{\frac{p}{p-q}[(N_i(x,y_{i+1}))^{\frac{p-q}{p}} - l_i^{\frac{p-q}{p}}] + C^{\frac{q-p}{p}}\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}\widetilde{f}(s,t)dsdt$$
$$+C^{\frac{q-p}{p}}\int_{\tau_2(y_i)}^{\tau_2(y)}\int_{\tau_1(x_i)}^{\tau_1(x)}\widetilde{g}(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\widetilde{h}(\xi,\eta)d\xi d\eta dsdt\} + l_i^{\frac{p-q}{p}}\}^{\frac{1}{p-q}}$$

holds for $(x,y) \in \Omega_{i+1,i+1},\ i = 0,1...,n$, (29)
where

$$\widetilde{f}(s,t) = f(\tau^{-1}(s),\tau^{-1}(t)),\ \widetilde{g}(s,t) = g(\tau^{-1}(s),\tau^{-1}(t)),\ \widetilde{h}(s,t) = h(\tau^{-1}(s),\tau^{-1}(t)),$$
$$N_{i-1}(x,y) = \widetilde{H}_{i-1}^{-1}\{C^{\frac{q-p}{p}}\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)}\widetilde{f}(s,t)dsdt$$
$$+C^{\frac{q-p}{p}}\int_{\tau_2(y_{i-1})}^{\tau_2(y)}\int_{\tau_1(x_{i-1})}^{\tau_1(x)}\widetilde{g}(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\widetilde{h}(\xi,\eta)d\xi d\eta dsdt,\ i = 1,2,...,n+1,$$
$$\widetilde{H}_i(t) = \widetilde{G}_i(2t - l_i) - \widetilde{G}_i(t),$$
$$\widetilde{G}_i(v) = \int_{l_i}^{v}\frac{1}{\omega(s^{\frac{1}{p}}(t))}ds,\ i = 0,1,...,n,$$
$$l_0 = 1,\ l_i = \gamma_i + \beta_i C^{\frac{1}{p}}\gamma_i^{\frac{1}{p}},\ i = 1,2,...,n,$$
$$\gamma_i = \widetilde{G}_{i-1}^{-1}\{\widetilde{G}_{i-1}(N_{i-1}(x_i,y_i)) + C^{\frac{q-p}{p}}\int_{\tau_2(y_{i-1})}^{\tau_2(y_i)}\int_{\tau_1(x_{i-1})}^{\tau_1(x_i)}\widetilde{f}(s,t)dsdt$$
$$+C^{\frac{q-p}{p}}\int_{\tau_2(y_{i-1})}^{\tau_2(y_i)}\int_{\tau_1(x_{i-1})}^{\tau_1(x_i)}\widetilde{g}(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\widetilde{h}(\xi,\eta)d\xi d\eta dsdt,\ i = 1,2,...,n.$$

Proof: From (27) and (28), for $(x,y) \in \Omega_{i,i},\ i = 1,2,...,n+1$ we have

$$|u^p(x,y)| \leq |\varphi(x,y)| + \int_{y_0}^{y}\int_{x_0}^{x}|M_1[s,t,u(\tau_1(s),\tau_2(t)),\int_{y_0}^{t}\int_{x_0}^{s}N_1(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta]|dsdt$$
$$+\int_{y_0}^{y_i}\int_{x_0}^{x}|M_2[s,t,u(\tau_1(s),\tau_2(t)),\int_{y_0}^{t}\int_{x_0}^{s}N_2(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta]|dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$
$$\leq C + \int_{y_0}^{y}\int_{x_0}^{x}|M_1[s,t,u(\tau_1(s),\tau_2(t)),\int_{y_0}^{t}\int_{x_0}^{s}N_1(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta]|dsdt$$
$$+\int_{y_0}^{y_i}\int_{x_0}^{x}|M_2[s,t,u(\tau_1(s),\tau_2(t)),\int_{y_0}^{t}\int_{x_0}^{s}N_2(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta]|dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$
$$\leq C + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)|\int_{y_0}^{t}\int_{x_0}^{s}N_1(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta|]dsdt$$
$$+\int_{y_0}^{y_i}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)|\int_{y_0}^{t}\int_{x_0}^{s}N_1(\xi,\eta,u(\tau_1(\xi),\tau_2(\eta)))d\xi d\eta|]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$

$$\leq C + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)|u(\tau_1(\xi),\tau_2(\eta))|^q d\xi d\eta]dsdt$$
$$+\int_{y_0}^{y_i}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)|u(\tau_1(\xi),\tau_2(\eta))|^q d\xi d\eta]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$
$$= C+\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)\int_{\tau_2(y_0)}^{\tau_2(t)}\int_{\tau_1(x_0)}^{\tau_1(s)}h(\tau_1^{-1}(\lambda),\tau_2^{-1}(\mu))|u(\lambda,\mu)|^q d\lambda d\mu]dsdt$$
$$+\int_{y_0}^{y_i}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)\int_{\tau_2(y_0)}^{\tau_2(t)}\int_{\tau_1(x_0)}^{\tau_1(s)}h(\tau_1^{-1}(\lambda),\tau_2^{-1}(\mu))|u(\lambda,\mu)|^q d\lambda d\mu]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$
$$= C+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[f(\tau_1^{-1}(\kappa),\tau_2^{-1}(\nu))|u(\kappa,\nu)|^q]d\kappa d\nu$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y)}[g(\tau_1^{-1}(\kappa),\tau_2^{-1}(\nu))\int_{\tau_2(y_0)}^{\nu}\int_{\tau_1(x_0)}^{\kappa}h(\tau_1^{-1}(\lambda),\tau_2^{-1}(\mu))|u(\lambda,\mu)|^q d\lambda d\mu]d\kappa d\nu$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_i)}\int_{\tau_1(x_0)}^{\tau_1(x)}[f(\tau_1^{-1}(\kappa),\tau_2^{-1}(\nu))|u(\kappa,\nu)|^q]d\kappa d\nu$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_i)}[g(\tau_1^{-1}(\kappa),\tau_2^{-1}(\nu))\int_{\tau_2(y_0)}^{\nu}\int_{\tau_1(x_0)}^{\kappa}h(\tau_1^{-1}(\lambda),\tau_2^{-1}(\mu))|u(\lambda,\mu)|^q d\lambda d\mu]d\kappa d\nu$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$
$$= C+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\widetilde{f}(\kappa,\nu)|u(\kappa,\nu)|^q + \widetilde{g}(\kappa,\nu)\int_{\tau_2(y_0)}^{\nu}\int_{\tau_1(x_0)}^{\kappa}\widetilde{h}(\lambda,\mu)|u(\lambda,\mu)|^q d\lambda d\mu]d\kappa d\nu$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_i)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\widetilde{f}(\kappa,\nu)|u(\kappa,\nu)|^q + \widetilde{g}(\kappa,\nu)\int_{\tau_2(y_0)}^{\nu}\int_{\tau_1(x_0)}^{\kappa}\widetilde{h}(\lambda,\mu)|u(\lambda,\mu)|^q d\lambda d\mu]d\kappa d\nu$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|$$
$$= C+\int_{\tau_2(y_0)}^{\tau_2(y)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\widetilde{f}(s,t)\omega(|u(s,t)|) + \widetilde{g}(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\widetilde{h}(\xi,\eta)\omega(|u(\xi,\eta)|)d\xi d\eta]dsdt$$
$$+\int_{\tau_2(y_0)}^{\tau_2(y_i)}\int_{\tau_1(x_0)}^{\tau_1(x)}[\widetilde{f}(s,t)\omega(|u(s,t)|) + \widetilde{g}(s,t)\int_{\tau_2(y_0)}^{t}\int_{\tau_1(x_0)}^{s}\widetilde{h}(\xi,\eta)\omega(|u(\xi,\eta)|)d\xi d\eta]dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\beta_j|u(x_j-0,y_j-0)|. \qquad (30)$$

On the other hand, we can easily deduce that $\widetilde{H}_i(t)$ are nondecreasing on $t \geq l_i,\ i = 0, 1, ..., n$. Moreover

$$\widetilde{G}_i(v) = \int_{l_i}^{v}\frac{1}{s^{\frac{q}{p}}(t)}ds = \frac{p}{p-q}(v^{\frac{p-q}{p}} - l_i^{\frac{p-q}{p}}),$$
$$i = 0, 1, ..., n, \qquad (31)$$

then after a suitable application of (31) and Theorem 1 we obtain the desired result.

4 Conclusions

In this paper, we have established a new generalized integral inequality with two independent variables for discontinuous functions. From the presented example one can see the established inequality is powerful in deriving explicit bounds for solutions of certain integral equations.

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