Two Nonlinear Long Wave Models in Shallow Water Generated by Applied Pressure

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Abstract

Water wave motion is described by the velocity potential for three dimensional viscous, incompressible and irrotational flow. Using dynamic and kinematic free surface conditions from Navier-Stokes equations, the nonlinear long wave models are generated by a disturbance moving at subcritical, critical and supercritical speed in unbounded shallow water. Nonlinearity (ε) and the dispersion (μ) are related as $\varepsilon = o(\mu^2)$, where nonlinearity is less than one. Then new forms of two long wave models are established in which nonlinear terms are expressed by the derivative of depth averaged velocity potential. The implements of the numerical algorithm are studied in the later section.

Keywords: Navier-Stokes equations; Linear and Non linear boundary conditions; Dimensional flow.

Introduction

High speed vessels are used widely as first means of transportation in waterways around the world. In restricted water, solitary wave can be generated ahead of the ship bow. First, Scott Russel discovered this phenomenon in 1834. Yile Li, Paul D. Sclavounos [1] investigated the nonlinear three dimensional upstream solitary long waves generated by a disturbance moving at sub critical, critical and supercritical speed in unbounded shallow water. They formulated the problem by a new modified generalized Boussinesq equation and solved numerically by an implicit finite difference algorithm. Craig and Nicholls [2] gave an analysis of travelling or progressive wave solutions to the problem of free surface water wave evolving under the influence of gravity in a fluid domain of infinite horizontal extent and of depth h, where $0 < h \leq \infty$. Robert Beck et al. [3] computed by two dimensional solitary waves generated by disturbances moving near the critical speed in shallow water by a time stepping procedure combined with a desingularized boundary integral method for irrotational flow. Guyenne and Grilli [4,5] performed to investigate the shoaling and breaking of a solitary wave over a sloping ridge with a lateral modulation in a three dimensional numerical wave tank. The numerical model solved fully nonlinear potential flow equations with a high order boundary element method combined with an explicit time integration method expressed in a mixed Eulerian-Lagrangian formulation. In a weakly nonlinear model equation for capillary-gravity water waves on a two dimensional free surface, Paul A. Milewski [6] showed numerically that there exist localized solitary travelling waves for a range of parameters spanning from the long wave limit to the wave-packet limit. Pelinovsky et al. [7] studied solitary wave where nonlinear shallow water theory was applied to obtain the analytical solution for the simplified bottom geometry. Wei-Ping Zhong et al. [8] investigated three dimensional spatiotemporal vector solitary waves in spherical coordinates and the exact three dimensional nonstationary solutions were obtained by the separation of variables and the Hirota bilinear method. W. Craig et al. [9] concerned the pairwise nonlinear interaction of solitary waves in the free surface of water lying over a horizontal bottom. Bai et al. [11] and Choi et al. [12] studied the nonlinear free surface flow produced by a three dimensional ship hull by means of the finite element method. Choi et al. [12] also reported the numerical results for a pressure distribution traveling at the critical speed in an open domain and found that the crestline of the leading soliton fits well with a parabola when the upstream wave develops. Casciola and Landrini [13] used an accurate boundary integral approach to simulate the flow and carried out a detailed comparison between the fully nonlinear model and generalized Boussinesq and forced KdV models. Ertekin et al. [14] pointed out that the blockage coefficient is the dominant parameter for the generation of solitons. Ertekin et al. [15] used the restricted Geen-Naghdi theory of fluid sheets to perform the three dimensional calculation of waves. Analyzing the linear dispersive relation near the critical speed, Katsis and Akylas [16] derived a forced nonlinear Kadomtsev-Petviashvili equation. Lee and Grimshaw [17] also employed the Kadomtsev-Petviashvili equation and reported various characteristics of upstream advancing waves in an open sea. In a joint numerical and experimental study, Lee et al. [18] found that both the generalized Boussinesq and forced

KdV models obtain qualitatively similar predictions of the phenomenon of the precursor solitons showing a satisfactory agreement with experiments. Michelle, H. Teng and Theodore, Y. Wu [19] examined the consistency and validity of the Boussinesq and KdV equations to describe nonlinear water wave generated by submerged disturbances moving with near critical speed in a rectangular channel. In this paper, two nonlinear long wave models are generated by the applied pressure.

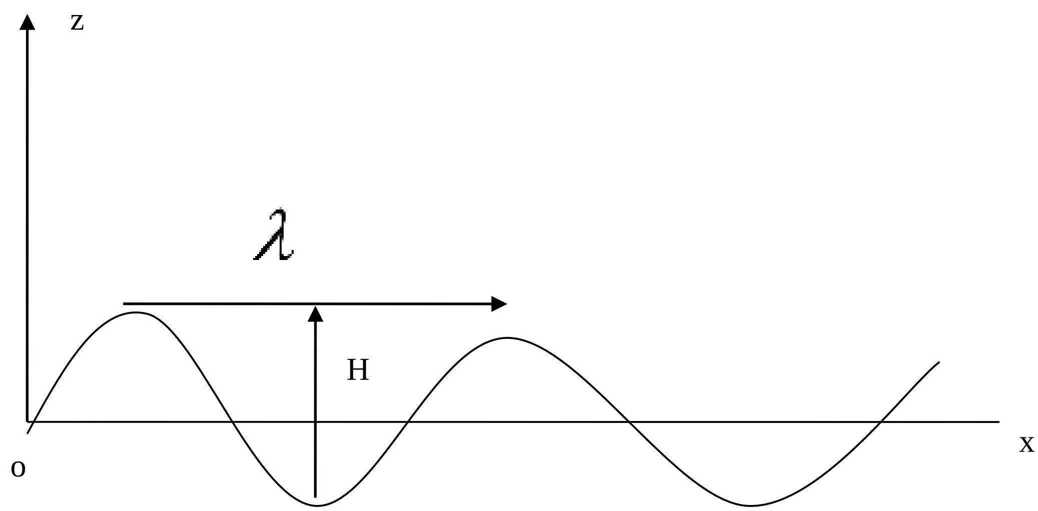

Figure: Wave train

Formulation

Under the assumption of incompressible and irrotational flow, the water wave motion is described by the velocity potential $\Phi(x, y, z, t)$ and the free surface water elevation $\eta(x, y, t)$. Taking fluid velocity $\underline{v} = \nabla\Phi$ and body force $\underline{F} = -\rho\underline{g}$, where $\underline{g}$ is the gravitational acceleration and $\underline{g} = -g\hat{k}$, Navier-Stokes equation becomes

$$\rho\left(\frac{\partial}{\partial t}(\nabla\Phi) + (\nabla\Phi.\nabla)\nabla\Phi\right) = -\rho g\hat{k} - \nabla p + \mu\nabla^2(\nabla\Phi) \qquad (1)$$

$$\therefore \int \rho\left(\frac{\partial}{\partial t}(\nabla\Phi) + (\nabla\Phi.\nabla)\nabla\Phi\right).d\underline{r} = \int\left(-\rho g\hat{k} - \nabla p + \mu\nabla^2(\nabla\Phi)\right)d\underline{r} \text{ (By integrating)}$$

$\therefore \frac{\partial\Phi}{\partial t}+\int(\nabla\Phi.\nabla)\nabla\Phi.d\underline{r}=-g\eta-\frac{p}{\rho}+\upsilon\nabla^2\Phi,\ at\ z=\eta,$ and $\frac{\mu}{\rho}=\upsilon$ is kinematic coefficient of viscosity. (2)

Here, $(\nabla\Phi.\nabla)\nabla\Phi=\frac{1}{2}\nabla.(\nabla\Phi)^2-\nabla\Phi\times(\nabla\times\nabla\Phi)$ (3)

For irrotational flow, the second term of Eq. (3) vanishes. So

$$(\nabla\Phi.\nabla)\nabla\Phi=\frac{1}{2}\nabla.(\nabla\Phi)^2$$

$$\therefore \int(\nabla\Phi.\nabla)\nabla\Phi.d\underline{r}=\frac{1}{2}\int\nabla.(\nabla\Phi)^2.d\underline{r}$$

$$=\frac{1}{2}(\nabla\Phi)^2$$

Substituting this value in Eq.(2), we have

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}(\nabla\Phi)^2=-g\eta-\frac{p}{\rho}+\upsilon\nabla^2\Phi,\ at\ z=\eta \quad (4)$$

Also the kinematic free surface boundary condition is

$$\frac{\partial\Phi}{\partial z}=\frac{\partial\eta}{\partial t}+\left(\frac{\partial\Phi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right)+\left(\frac{\partial\Phi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right)\ at\ z=\eta \quad (5)$$

And the bottom boundary condition is

$$\frac{\partial\Phi}{\partial z}=0,\ at\,z=-h \quad (6)$$

Let a pressure distribution advance at the constant speed U acting on the surface of a layer of water with uniform depth h. Steady current is moving in the positive x-direction with speed U (Fig. 1).

In this reference frame, the velocity potential $\Phi(x,y,z,t)$ can be decomposed as

$$\Phi(x,y,z,t)=\phi(x,y,z,t)+Ux \quad (7)$$

In which $\phi(x,y,z,t)$ is the disturbance velocity potential representing the flow motion induced by the pressure distribution on the free surface.

The velocity potential $\phi(x,y,z,t)$ satisfies the Laplace equation

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \quad (8)$$

Substituting Eq. (7) in Eqs. (4), (5), and (6), we have

$$\phi_t + \frac{1}{2}(\nabla\phi).(\nabla\phi) + U\phi_x + g\eta + \frac{p}{\rho} - \upsilon\nabla^2\phi = 0, \ at \ z = \eta \qquad (9)$$

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y + U\eta_x = \phi_z \ at \ z = \eta \qquad (10)$$

and

$$\phi_z = 0 \ at\, z = -h \qquad (11)$$

Assuming wave amplitude a, the characteristic wave number k and the characteristic horizontal velocity $\sqrt{gh}$ where h is the vertical scale .In dimensionless form, the above variables are as follows:

$$x = \frac{x'}{k}, y = \frac{y'}{k}, \eta = a\eta', z = hz'$$

$$\phi = \frac{a}{k}\sqrt{\frac{g}{h}}\phi', t = \frac{1}{k\sqrt{gh}}t', p = a\rho g p' \ and \ \upsilon = h\sqrt{gh}\upsilon'$$

And the dominant parameters are

$\varepsilon = \frac{a}{h}, \mu = kh$ and the depth Froude number is $F_h = \frac{U}{\sqrt{gh}}$.

With these dimensionless variables, Eqs. (8), (9),(10) and (11) become

$$\left(\phi'_{x'x'} + \phi'_{y'y'}\right) + \frac{1}{\mu^2}\phi'_{z'z'} = 0 \qquad (12)$$

$$\phi'_{t'} + \frac{\varepsilon}{2}\left[\left(\phi'^{2}_{x'} + \phi'^{2}_{y'}\right) + \frac{1}{\mu^2}\phi'^{2}_{z'}\right] + F_h\phi'_{x'} + \eta' + p' - \upsilon'\mu\left(\left(\phi'_{x'x'} + \phi'_{y'y'}\right) + \frac{1}{\mu^2}\phi'_{z'z'}\right) = 0 \qquad (13)$$

$$\eta'_{t'} + \varepsilon\left(\phi'_{x'}\eta'_{x'} + \phi'_{y'}\eta'_{y'}\right) + F_h\eta'_{x'} = \frac{1}{\mu^2}\phi'_{z'} \ at \ z' = \varepsilon\eta' \qquad (14)$$

and

$$\phi'_{z'} = 0, at \ z' = -1 \qquad (15)$$

For the convenience of our calculation, we drop primes from Eqs. (12) to (15),

$$\left(\phi_{xx} + \phi_{yy}\right) + \frac{1}{\mu^2}\phi_{zz} = 0 \qquad (16)$$

$$\phi_t + \frac{\varepsilon}{2}\left[\left(\phi_x^2 + \phi_y^2\right) + \frac{1}{\mu^2}\phi_z^2\right] + F_h\phi_x + \eta + p - \upsilon\mu\left(\left(\phi_{xx} + \phi_{yy}\right) + \frac{1}{\mu^2}\phi_{zz}\right) = 0 \qquad (17)$$

$$\eta_t + \varepsilon(\phi_x \eta_x + \phi_y \eta_y) + F_h \eta_x = \frac{1}{\mu^2}\phi_z \ at \ z = \varepsilon\eta \tag{18}$$

and

$$\phi_z = 0, \ at \ z = -1 \tag{19}$$

Expanding the velocity potential $\phi(x, y, z, t)$ in power series with respect to the vertical coordinate about $z = -1$.

$$\therefore \phi(x, y, z, t) = \sum_{n=0}^{\infty} (z+1)^n \phi_n(x, y, t) \tag{20}$$

Substituting this value in equation (16), we get

$$\mu^2(\phi_{xx} + \phi_{yy}) + \phi_{zz} == \sum_{n=0}^{\infty} (z+1)^n \{(n+1)(n+2)\phi_{n+2} + \mu^2 \nabla^2 \phi_n\} \tag{21}$$

In the range $[-1, \varepsilon\eta]$, z is arbitrary and the coefficients of the power of $z+1$ must vanish to satisfy Eq.(21), thus we have,

$$(n+1)(n+2)\phi_{n+2} + \mu^2 \nabla^2 \phi_n = 0, n = 0,1,2,.................... \tag{22}$$

From the recursive relation (22), the velocity potential components with odd subscripts all vanish, i.e.,

$$\phi_1 = \phi_3 = ...=, \phi_{2m+1} = ... = 0, \ m = 0,1,2,..................... \tag{23}$$

Therefore, from Eq. (22), the velocity potential components of even term with ϕ_0, the zero order terms are as follows:

$$\phi_{2m} = -\frac{\mu^2}{2m(2m-1)} \nabla^2 \phi_{2m-2} \tag{24}$$

From Eq. (20),

$$\phi(x, y, z, t) = \phi_0 + (z+1)^2 \phi_2(x, y, t) + (z+1)^4 \phi_4(x, y, t) + ... \tag{25}$$

Substituting Eq.(24) in Eq.(25), we have

$$\phi(x, y, z, t) = \phi_0 - \frac{\mu^2}{2}(z+1)^2 \nabla^2 \phi_0 + \frac{\mu^4}{24}(z+1)^4 \nabla^2 \nabla^2 \phi_0 + o(\mu^6) \tag{26}$$

Depth averaged velocity potential is defined by

$$\bar{\phi}(x,y,t)=\frac{1}{1+\varepsilon\eta}\int_{-1}^{\varepsilon\zeta}\phi(x,y,z,t)dz$$
$$=\phi_0-\frac{\mu^2}{6}(1+\varepsilon\eta)^2\nabla^2\phi_0+o(\mu^4) \tag{27}$$

Therefore,

$$\phi_0(x,y,t)=\bar{\phi}(x,y,t)+\frac{\mu^2H^2}{6}\nabla^2\bar{\phi}+o(\mu^4),\ \text{where}\ H=1+\varepsilon\eta \tag{28}$$

Substituting Eq. (28) in Eq. (26) and using $z=1+\varepsilon\eta$, the two dimensional velocity potential

$$\phi(x,y,t)=\bar{\phi}+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}-\mu^4\left(\frac{(1+\varepsilon\eta)^2}{12}\nabla^2\left(H^2\nabla^2\bar{\phi}\right)-\frac{1}{24}(1+\varepsilon\eta)^4\nabla^2\nabla^2\bar{\phi}\right)+o(\mu^6) \tag{29}$$

For shallow water waves, the nonlinearity ε and dispersion μ are related as follows

$$\varepsilon=o(\mu^2) \tag{30}$$

The primary time variable is taken as slow, then we have

$$\varepsilon^2<o\left(\frac{\partial}{\partial t}\right)<1$$

$$\text{Therefore,}\ o\left(\varepsilon\frac{\partial}{\partial t}\right)<<o(\varepsilon),\qquad o\left(\mu^2\frac{\partial}{\partial t}\right)<<o(\mu^2). \tag{31}$$

Using the above assumption and substituting Eq. (29) into Eq. (17) becomes

$$\bar{\phi}_t+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}_t+F_h\left\{\bar{\phi}_x+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}_x\right\}+\eta$$
$$+\frac{\varepsilon}{2}\left[\begin{array}{l}\left\{\bar{\phi}_x+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}_x\right\}^2\\+\left\{\bar{\phi}_y+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}_y\right\}^2\end{array}\right]$$
$$-p-\mu\upsilon\left[\bar{\phi}_{xx}+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}_{xx}+\bar{\phi}_{yy}+\mu^2\left(\frac{H^2}{6}-\frac{(1+\varepsilon\eta)^2}{2}\right)\nabla^2\bar{\phi}_{yy}\right]+o(\mu^4)=0$$

where time derivative of $o(\mu^2)$ and $o(\varepsilon,\mu^2)$ is omitted and also $H=1+\varepsilon\eta$, then above equation can be written as

$$\bar{\phi}_t + F_h\bar{\phi}_x = -\eta - \frac{\varepsilon}{2}\left(\bar{\phi}_x{}^2 + \bar{\phi}_y{}^2\right) - p + \frac{1}{3}\mu^2 F_h \nabla^2\bar{\phi}_x + \mu\upsilon\left(\bar{\phi}_{xx} + \bar{\phi}_{yy}\right) \tag{32}$$

Similarly, from Eq. (18), we have

$$\eta_t + F_h\eta_x = -\nabla.\left\{(1+\varepsilon\eta)\left(\nabla\bar{\phi}\right)\right\} + o\left(\mu^4\right) \tag{33}$$

The free surface elevation η can be expressed as in terms of $\bar{\phi}$ explicitly. From Eq. (32), we have

$$\eta = -\bar{\phi}_t - F_h\bar{\phi}_x - \frac{\varepsilon}{2}\left(\bar{\phi}_x{}^2 + \bar{\phi}_y{}^2\right) - p + \frac{1}{3}\mu^2 F_h \nabla^2\bar{\phi}_x + \mu\upsilon\left(\bar{\phi}_{xx} + \bar{\phi}_{yy}\right) \tag{34}$$

Substituting the value of Eq. (34) in Eq. (33) and we have

$$\eta_t + F_h\eta_x = -\nabla.\left\{\left[1+\varepsilon\left\{-\bar{\phi}_t - F_h\bar{\phi}_x - \frac{\varepsilon}{2}\left(\bar{\phi}_x{}^2 + \bar{\phi}_y{}^2\right) - p + \frac{1}{3}\mu^2 F_h\nabla^2\bar{\phi}_x + \mu\upsilon\left(\bar{\phi}_{xx} + \bar{\phi}_{yy}\right)\right\}\right]\left(\nabla\bar{\phi}\right)\right\}$$
$$+ o\left(\mu^4\right)$$

Neglecting the order higher than $o\left(\varepsilon,\mu^2\right)$ and time derivative of $o(\varepsilon)$ or $o\left(\mu^2\right)$,

$$\therefore \eta_t + F_h\eta_x = -\nabla^2\bar{\phi} + \left[\varepsilon\left\{\begin{array}{l} F_h\left(2\bar{\phi}_x\bar{\phi}_{xx} + \bar{\phi}_{xy}\bar{\phi}_y + \bar{\phi}_x\bar{\phi}_{yy}\right) - \mu\left(\bar{\phi}_{xxx}\bar{\phi}_x + \bar{\phi}_{xx}{}^2 + \bar{\phi}_{xyy}\bar{\phi}_x + \bar{\phi}_{xx}\bar{\phi}_{yy}\right) \\ -\mu\upsilon\left(\bar{\phi}_{xxy}\bar{\phi}_y + \bar{\phi}_{xx}\bar{\phi}_{yy} + \bar{\phi}_{yyy}\bar{\phi}_y + \bar{\phi}_{yy}{}^2\right) \end{array}\right\}\right] + \varepsilon\nabla.\left(p\nabla\bar{\phi}\right) \tag{35}$$

Eqs. (34) and (35) are two forms of nonlinear long wave models in terms of depth averaged velocity potential.

Numerical Algorithm

The implements of the numerical algorithm are described in this section. The unknown averaged velocity potential $\left(\bar{\phi}\right)$ and the free surface elevation $\left(\bar{\eta}\right)$ in the computational domain at the $(n+1)$th time level satisfy the Eqs. (34) and (35). In this paper, we develop an implicit finite difference algorithm to study the Eqs. (34) and (35). The discretized finite difference equations can be written as

$$(\eta)_{i,j}^{n+1} = -\left(\bar{\phi}_t\right)_{i,j}^{n+1} - F_h\left(\bar{\phi}_x\right)_{i,j}^{n+1} - \frac{\varepsilon}{2}\left(\bar{\phi}_x{}^2 + \bar{\phi}_y{}^2\right)_{i,j}^{n+1} - p_{i,j} + \frac{1}{3}\mu^2 F_h\left(\bar{\phi}_{xxx} + \bar{\phi}_{xyy}\right)_{i,j}^{n+1} + \mu\upsilon\left(\bar{\phi}_{xx} + \bar{\phi}_{yy}\right)_{i,j}^{n+1}$$

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(36)

$$(\eta_t)_{i,j}^{n+1} + F_h(\eta_x)_{i,j}^{n+1} = -(\bar{\phi}_{xx} + \bar{\phi}_{yy})_{i,j}^{n+1} - \varepsilon \left\{ \begin{aligned} & F_h\left(2\bar{\phi}_x\bar{\phi}_{xx} + \bar{\phi}_{xy}\bar{\phi}_y + \bar{\phi}_x\bar{\phi}_{yy}\right)_{i,j}^{n+1} - \mu\left(\bar{\phi}_{xxx}\bar{\phi}_x + \bar{\phi}_{xx}{}^2 + \bar{\phi}_{xyy}\bar{\phi}_x + \bar{\phi}_{xx}\bar{\phi}_{yy}\right)_{i,j}^{n+1} \\ & - \mu\upsilon\left(\bar{\phi}_{xx}\bar{\phi}_{yy} + \bar{\phi}_{xxy}\bar{\phi}_y + \bar{\phi}_{yyy}\bar{\phi}_y + \bar{\phi}_{yy}{}^2\right)_{i,j}^{n+1} \end{aligned} \right\} \quad (37)$$

The adopted finite difference scheme (36), (37) is of fully implicit type and is unconditionally stable. The principal forcing term of the formation of the nonlinear long waves due to the applied pressure distribution p on right hand side of Eq. (34) and the last term involving pressure of right hand side of Eq. (35) is $o(\varepsilon)$ and we omit it.

The finite difference for the derivatives have been taken as

$$(\phi_x)_i^j = \left(\frac{\partial\phi}{\partial x}\right)_i^j = \frac{\phi_{i+1}^{j+1} - \phi_{i-1}^{j+1}}{2\Delta x} = \frac{\phi_{i+1}^{j+1} - \phi_i^{j+1}}{\Delta x} = \frac{3\phi_i^{j+1} - 4\phi_{i-1}^{j+1} + \phi_{i-2}^{j+1}}{2\Delta x}$$

$$(\phi_{xx})_i^j = \left(\frac{\partial^2\phi}{\partial x^2}\right)_i^j = \frac{\phi_{i+1}^{j+1} - 2\phi_i^{j+1} + \phi_{i-1}^{j+1}}{(\Delta x)^2}$$

$$(\phi_{xxx})_i^j = \left(\frac{\partial^3\phi}{\partial x^3}\right)_i^j = \frac{\phi_{i+2}^{j+1} - 2\phi_{i+1}^{j+1} + 2\phi_{i-1}^{j+1} - \phi_{i-2}^{j+1}}{2(\Delta x)^3}$$

$$(\phi_{xy})_i^j = \left(\frac{\partial^2\phi}{\partial x\partial y}\right)_i^j = \frac{\phi_{i+1}^{j+1} - \phi_{i+1}^{j-1} - \phi_{i-1}^{j+1} + \phi_{i-1}^{j-1}}{4\Delta x\Delta y}$$

$$(\phi_{xxy})_i^j = \left(\frac{\partial^3\phi}{\partial x^2\partial y}\right)_i^j = \frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_i^j = \frac{\phi_{i+2}^{j+1} - \phi_{i+2}^{j-1} - 2\phi_i^{j+1} + 2\phi_i^{j-1} + \phi_{i-2}^{j+1} - \phi_{i-2}^{j-1}}{8(\Delta x)^2\Delta y}$$

$$(\phi_{xyy})_i^j = \left(\frac{\partial^3\phi}{\partial x\partial y^2}\right)_i^j = \frac{\phi_{i+1}^{j+2} - \phi_{i-1}^{j+2} - 2\phi_{i+1}^{j} + 2\phi_{i-1}^{j} + \phi_{i+1}^{j-2} - \phi_{i-1}^{j-2}}{8\Delta x(\Delta y)^2}$$

The three time level scheme is used to approximate the time derivative

$$(\bar{\phi}_t)_{i,j}^{n+1} = \frac{3\bar{\phi}_{i,j}^{n+1} - 4\bar{\phi}_{i,j}^{n} + \bar{\phi}_{i,j}^{n-1}}{2\Delta t}$$

$$(\eta_t)_{i,j}^{n+1} = \frac{3\eta_{i,j}^{n+1} - 4\eta_{i,j}^{n} + \eta_{i,j}^{n-1}}{2\Delta t}$$

The solution of the nonlinear Eqs. (36) and (37) can be obtained iteratively. The initial value of the variables at the next time step is taken as the value at the nth step,

$\overline{\phi}_{i,j}^{n+1,0} = \overline{\phi}_{i,j}^{n}$, $\eta_{i,j}^{n+1,0} = \eta_{i,j}^{n}$, in which the second superscript indicates the index of iteration. Hence, at each iterative step k, Eqs. (36) and (37) become

$$\frac{3\overline{\phi}_{i,j}^{n+1,k} - 4\overline{\phi}_{i,j}^{n} + \overline{\phi}_{i,j}^{n-1}}{2\Delta t} + F_h \frac{\overline{\phi}_{i+1,j}^{n+1,k} - \overline{\phi}_{i-1,j}^{n+1,k}}{2\Delta x} - \frac{\mu^2 F_h}{3}\left(\frac{\overline{\phi}_{i+2,j}^{n+1,k} - 2\overline{\phi}_{i+1,j}^{n+1,k} + 2\overline{\phi}_{i-1,j}^{n+1,k} - \overline{\phi}_{i-2,j}^{n+1,k}}{2(\Delta x)^3}\right)$$
$$= -\eta_{i,j}^{n+1,k} + \frac{\mu^2 F_h}{3}\left(\overline{\phi}_{xyy}\right)_{i,j}^{n+1,k'} - \frac{\varepsilon}{2}\left(\overline{\phi}_x^{\,2} + \overline{\phi}_y^{\,2}\right)_{i,j}^{n+1,k'} - p_{i,j} + \mu\upsilon\left(\overline{\phi}_{xx} + \overline{\phi}_{yy}\right)_{i,j}^{n+1,k'} \quad (38)$$

and

$$\frac{3\eta_{i,j}^{n+1,k} - 4\eta_{i,j}^{n} + \eta_{i,j}^{n-1}}{2\Delta t} + F_h \frac{\eta_{i+1,j}^{n+1,k} - \eta_{i-1,j}^{n+1,k}}{2\Delta x} = -\left(\overline{\phi}_{xx} + \overline{\phi}_{yy}\right)_{i,j}^{n+1,k'}$$
$$-\varepsilon\left\{\begin{array}{l} F_h\left(2\overline{\phi}_x\overline{\phi}_{xx} + \overline{\phi}_{xy}\overline{\phi}_y + \overline{\phi}_x\overline{\phi}_{yy}\right)_{i,j}^{n+1,k'} \\ +\mu\upsilon\left(\begin{array}{l}\overline{\phi}_{xxx}\overline{\phi}_x + \overline{\phi}_{xx}^{\,2} + \overline{\phi}_{xyy}\overline{\phi}_x \\ +2\overline{\phi}_{yy}\overline{\phi}_{xx} + \overline{\phi}_{xxy}\overline{\phi}_y + \overline{\phi}_{yyy}\overline{\phi}_y + \overline{\phi}_{yy}^{\,2}\end{array}\right)_{i,j}^{n+1,k'}\end{array}\right\} \quad (39)$$

in which $\eta_{i,j}^{n+1,k}$ and $\overline{\phi}_{i,j}^{n+1,k}$ are the prediction values of the $\eta_{i,j}$ and $\overline{\phi}_{i,j}$ after the k-th iteration. The superscript k' is used in the terms of the right hand side since both the values at the $(k-1)$th and kth iterations are used to evaluate the derivatives.

Conclusion

For viscous incompressible and irrotational flow, water wave motion is described for two forms of nonlinear long wave models. Using dynamic and kinematic free surface conditions taken from Navier-Stokes equation, the water wave problem is formulated on the bottom boundary condition. Expressing velocity potential in power series with respect to the vertical coordinates, velocity potential of even order is derived. Using the relations $\varepsilon = o(\mu^2)$ and $\varepsilon < o\left(\frac{\partial}{\partial t}\right) < 1$, two forms of long wave models in which nonlinear terms are expressed by the derivative of depth averaged velocity potential $\overline{\phi}$. Here, the mathematical model and numerical schemes are described and these are applied to simulate the nonlinear long waves induced by a pressure distribution in shallow water. In absence of viscous term and dispersion term, our model becomes a modified generalized

Boussinesq equations which are obtained in paper [1]. Then the implements of the numerical algorithm are established.

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