On a trigonometric inequality of Askey and Steinig.

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Abstract. A short proof is given for the inequality

$$\frac{d}{d\theta}\sum_{k=1}^{n}\frac{\sin k\theta}{k\sin\theta/2} < 0 \qquad \text{for } 0<\theta<\pi,$$

supplemented by a discussion of some related results.

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1. Motivation and results.

Let the function f_n on $]0,\pi]$ for $n \in I\!N$ be defined by

$$f_n(\theta) := \sum_{k=1}^{n}\frac{\sin k\theta}{k\sin\theta/2} \tag{1}$$

In [**1**] Askey and Steinig established the inequality

$$\frac{d}{d\theta}f_n(\theta) < 0 \qquad \text{for } 0<\theta<\pi. \tag{2}$$

Since $f_n(\pi) = 0$ this inequality implies

$$\sum_{k=1}^{n}\frac{\sin k\theta}{k} > 0 \qquad \text{for } 0<\theta<\pi, \tag{3}$$

an inequality conjectured 1910 by Fejér and proved by Jackson [**4**], Gronwall [**3**], Fejér [**2**], Landau [**5**] (also reproduced in [**7**,II.9.4]) and Turán [**6**]. Askey's and Steinig's proof of (2) is based on (3) and on careful estimates of various trigonometric sums in certain subintervals of $]0,\pi]$. The purpose of this note is to give a comparatively simple proof of (2) and to point out some conclusions which add to motivate interest in this inequality.

2. Proof of the inequality.

It seems convenient to introduce the functions g and h_n defined on $[0,\pi]$ by

$$g(\theta) := \frac{\sin\theta/2}{\theta/2} \qquad \text{for } 0<\theta\le\pi,$$

$$
\begin{aligned}
g(0) &= 1, \\
h_n(\theta) &:= \sum_{k=1}^{n} \frac{\sin k\theta}{k\theta} \qquad \text{for } 0 < \theta \le \pi, \\
h_n(0) &:= n.
\end{aligned}
$$

Since $f_n = 2h_n/g$ inequality (2) holds if and only if each of the following inequalities holds on $]0,\pi[$:

$$
\begin{aligned}
g(\theta)\cdot h_n'(\theta) &< g'(\theta)\cdot h_n(\theta) \\
\frac{h_n'}{h_n}(\theta) &< \frac{g'}{g}(\theta) \qquad \text{because of (3)} \qquad (4) \\
\log h_n(\theta) - \log n &< \log g(\theta) \qquad \text{(integrating (4) from 0 to } \theta) \\
\frac{1}{n}\sum_{k=1}^{n} \frac{\sin k\theta}{k} &< 2\sin\theta/2 \qquad (5)
\end{aligned}
$$

The last inequality obviously holds for $\frac{\pi}{2} \le \theta \le \pi$ since there one has

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\sin k\theta}{k} \le 1 < \sqrt{2} = 2\sin\pi/4 \le 2\sin\theta/2.$$

It remains to check (5) on $]0,\pi/2[$. There, since $\cos\theta > 0$, it may readily be shown by induction that

$$\sin k\theta \le k\sin\theta$$

which implies

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\sin k\theta}{k} \le \sin\theta = 2\sin\theta/2\cos\theta/2 < 2\sin\theta/2 \qquad \square$$

3. Additional remarks.

1) Askey and Steinig mention that (3) implies the following observation due to J.Burtoz: for $z \in]-1,1[$, $z \ne 0$ and $n \in \mathbb{N}$ one has

$$\sum_{k=1}^{n} z^{k-1}\frac{\sin k\theta}{k\sin\theta} \ne 0 \qquad \text{for all } \theta \in \mathbb{R}.$$

This assertion may be generalized in the following way:
If $a_1 \ge a_2 \ge \cdots \ge a_n > 0$, then the function p_n defined on $\mathbb{R}$ by

$$
\begin{aligned}
p_n(\theta) &= \sum_{k=1}^{n} a_k \frac{\sin k\theta}{k\sin\theta} \qquad \text{for } \theta \ne m\pi,\ m \in \mathbb{Z} \\
p_n(2m\pi) &= \sum_{k=1}^{n} a_k \\
p_n((2m+1)\pi) &= \sum_{k=1}^{n} (-1)^{k-1} a_k
\end{aligned}
$$

satisfies

$$\sum_{k=1}^{n}(-1)^{k-1}a_k\frac{\sin k\theta}{k\sin\theta}=p_n(\theta+\pi) \qquad (6)$$

and is positive for all $\theta\neq\pi+2m\pi$, *except in* $\theta=\pi+2m\pi$ *if* $n\equiv 0 \pmod 2$ *and* $a_{2k-1}=a_{2k}$ $(1\leq k\leq\frac{n}{2})$.

The function p_n is readily seen to be even, periodic with period 2π, continuous on $\mathbb{R}$, and to satisfy (6). Positivity for $0<\theta<\pi$ may be shown by induction: for $n=1$ the assertion is trivial; for $n>1$ one has

$$p_n(\theta) = a_n\sum_{k=1}^{n}\frac{\sin k\theta}{k\sin\theta}+\sum_{k=1}^{n-1}(a_k-a_n)\frac{\sin k\theta}{k\sin\theta}>0$$

since the first term on the right side is positive and the second one is non-negative by inductive hypothesis. For $\theta=0$ and for $\theta=\pi$ the asssertions are clear.

2) Inequality (2) also furnishes some information concerning the Dirichlet-kernel D_n defined by

$$\begin{aligned} D_n(\theta) &= \frac{1}{2}+\sum_{k=1}^{n}\cos k\theta\left(=\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right) && 0<\theta\leq\pi \\ D_n(0) &= n+\frac{1}{2} \end{aligned}$$

a) The corresponding mean value function

$$\begin{aligned} M_n(\theta) &= \frac{1}{\theta}\int_0^{\theta}D_n(t)\,dt=\frac{1}{2}+\sum_{k=1}^{n}\frac{\sin k\theta}{k\theta} && 0<\theta\leq\pi \\ M_n(0) &= n+\frac{1}{2} \end{aligned}$$

is also monotonically decreasing on $[0,\pi]$

b)

$$\sum_{k=1}^{n}\cos k\theta<\sum_{k=1}^{n}\frac{\sin k\theta}{k\theta} \qquad 0<\theta<\pi$$

c)

$$D_n(\theta)<M_n(\theta) \qquad 0<\theta<\pi.$$

In fact,

$$\sum_{k=1}^{n}\frac{\sin k\theta}{k\theta}=\frac{1}{2}\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\sum_{k=1}^{n}\frac{\sin k\theta}{k\sin\frac{\theta}{2}}$$

by (2) is a product of two monotonically decreasing functions on $]0, \pi]$. This again already implies

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k\theta} \geq \sum_{k=1}^{n} \frac{\sin k\pi}{k\pi} = 0.$$

Assertion a)

$$\frac{d}{d\theta} M_n(\theta) = \frac{1}{\theta} \sum_{k=1}^{n} \cos k\theta - \frac{1}{\theta^2} \sum_{k=1}^{n} \frac{\sin k\theta}{k} < 0 \qquad 0 < \theta < \pi$$

is equivalent with

$$\sum_{k=1}^{n} \cos k\theta < \sum_{k=1}^{n} \frac{\sin k\theta}{k\theta} \qquad 0 < \theta < \pi$$

This again is equivalent with

$$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta < \frac{1}{2} + \sum \frac{\sin k\theta}{k\theta} = M_n(\theta) \qquad 0 < \theta < \pi.$$

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