Triangular Numbers in Quadratic Functions Form, Generating Functions and Continued Fractions

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ABSTRACT

The nth triangular number denoted by T_n is defined as the sum of the first n consecutive positive integers, and a positive integer n is a triangular number if and only if $T_n = \frac{n(n+1)}{2}$. In this paper we represent a triangular number by a quadratic function i.e., for each $m \in \mathbb{Z}$ the necessary and sufficient condition for a quadratic function $f(x) = x^2 + x - 2m$ to be triangular is proved. We also prove, a theorem associated to a rational root d of a quadratic function $f(x)$ to be a triangular number T_n. We also use Generating function to represent the sets of Quotients of triangular numbers.

KEYWORDS: Triangular Numbers, Quadratic functions, Sequences and Factorials

INTRODUCTION

A triangular number T_n is a number of the form $T_n = 1 + 2 + 3 + \cdots + n$, where n is a natural number. For instance, the first few triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45 [1, 2, 3]. A well known fact about triangular numbers is that y is a triangular number if and only if $(8y + 1)$ is a perfect square. Triangular numbers can be thought of as the numbers of dots that can be arranged in the shape of a triangle. Another interesting aspect of the triangular numbers is that they are in consecutive pairs of alternating odd and even integers. The table of triangular numbers (pages 6 and 7) illustrates this fact.

Lemma 0.0.1: *A positive integer k is called Triangular if and only if there exists a positive integer n such that $\boldsymbol{k = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = T_n}$ [1, 4, 5, 6].*

Example 0.0.2 *Prove that $25k + 3$ is triangular if k is triangular.*

Proof: *We show that $25k + 3 = \frac{x(x+1)}{2}$ for some $x \geq 1$. Suppose k is triangular. By (Lemma 0.0.1), for some $n \geq 1$, $k = \frac{n(n+1)}{2}$. Hence, $25k + 3 = 25\left(\frac{n(n+1)}{2}\right) + 3 = \frac{(25n^2 + 25n + 6)}{2} = \frac{(5n+2)(5n+3)}{2}$. Set $x = 5n + 2$. Then $5n + 3 = x + 1$ and $25k + 3 = \frac{(5n+2)(5n+3)}{2} = \frac{x(x+1)}{2}$. Therefore $25k + 3$ is triangular.* ∎

Theorem 0.0.3 A positive integer m is a triangular number if and only if an odd root d of a quadratic function $f(x) = x^2 + x - 2m$ divides m.

Proof: *($\Rightarrow$) Suppose a positive integer m is triangular and d is an odd root of $f(x)$. We show that $d | m$. There exists $n \in \mathbb{Z}^+$ such that $m = \frac{n(n+1)}{2}$ (Lemma 0.0.1). This implies $f(x) = x^2 + x - 2 = x^2 + x - 2\frac{n(n+1)}{2} = x^2 + x - n(n+1)$. Because d is a root of $f(x)$ we have $f(d) = d^2 + d - n(n+1) = 0$*

Using quadratic formula, we have $d = \frac{-1 \pm \sqrt{1^2 - 4(1)(n(n+1))}}{2} = \frac{-1 \pm \sqrt{1+4n^2+4n}}{2}$
$= \frac{-1 \pm \sqrt{(2n+1)^2}}{2} = \frac{-1 \pm |2n+1|}{2}$,

This implies $d = \frac{-1+(2n+1)}{2}$, *or* $d = \frac{-1-(2n+1)}{2}$ *that is,* $d = n$ *or* $d = -(n+1)$.
We consider two cases. First $m = \frac{n(n+1)}{2}$ *when n is even i.e.,* $n = 2k$ *for some* $k \in \mathbb{Z}^+$. *This implies* $m = \frac{2k(2k+1)}{2} = k(2k+1)$ *, and then* $d = -(n+1) = -(2k+1)|m$. *Second when n is odd i.e.,* $n = 2k+1$ *for some* $k \in \mathbb{Z}^+$. *We have* $m = \frac{(2k+1)(2k+2)}{2} = (2k+1)(k+1)$ *and* $d = n = (2k+1)|m$.

($\Leftarrow$) *Suppose an odd root* d *of* $f(x) = x^2 + x - 2m$ *divides* m. *We show that* m *is triangular. As* d *is a root of* $f(x) = x^2 + x - 2m$ *it follows* $f(d) = d^2 + d - 2m = 0$ *, and* d *divides* m *implies* $m = dc$ *for some* $c \in \mathbb{Z}^+$. *Combining the former and later we have*

$$f(d) = d^2 + d - 2(dc) = 0.$$

Therefore, $d^2 + d - 2(dc) = d(d+1-2c) = 0$, *and either* $d = 0$ *or* $(d+1-2c) = 0$. *As* d *divides* m, $d \neq 0$. *This implies that* $(d+1-2c) = 0$, *and* $2c = d+1$, *and* $c = \frac{d+1}{2}$. *Thus,* $m = dc = \frac{d(d+1)}{2}$ *and hence* m *is triangular.* ∎

Theorem 0.0.4 *All roots of a quadratic function* $f(x) = x^2 + x - 2m$ *are rational if and only if* m *is triangular.*

Proof. ($\Rightarrow$) *Suppose a quadratic function* $f(x) = x^2 + x - 2m$ *has rational root* d. *Then the root* $d = \frac{-1 \pm \sqrt{1+8m}}{2}$ *is rational. This implies the discriminant* $D = (1 + 8m)$ *must be a perfect square. If* $(1 + 8m)$ *is a perfect square, then there exists an integer* p *such that* $p^2 = 1 + 8m$. *But* $1 + 8m = 1 + 2(4m) = 1 + 2t$ *for some* $t = 4m \in \mathbb{Z}^+$ *and is an odd integer. Consequently* p^2 *is odd and* p *is odd too. This implies there is* $a \in \mathbb{Z}^+$ *such that* $p = 2a + 1$ *and* $(2a+1)^2 = 1 + 8m$. *Hence* $4a^2 + 4a + 1 = 1 + 8m$ *and* $4a^2 + 4a = 8m$. *Consequently,* $4a(a+1) = 8m$ *and then* $m = \frac{a(a+1)}{2}$. *Therefore* m *is a triangular number.*

Suppose m *is a triangular number and a quadratic function* $f(x) = x^2 + x - 2m$ *has a real root. Then we show that it is not an irrational number. The quadratic function* $f(x) = x^2 + x - t(t+1)$ *where* $m = \frac{t(t+1)}{2}$ *is triangular implies* $f(x) = x^2 + x - t(t+1) = (x-t)(x+t+1) = 0$ *has a root* x *where either* $x = t$ *or* $x = -(t+1)$ *which is a rational number.* ∎

Corollary 0.0.5 *If a quadratic function* $f(x) = x^2 + x - 2m$ *has a root* t, *then* $m = \frac{t(t+1)}{2}$.

Proof: *Suppose a quadratic function* $f(x) = x^2 + x - 2m$ *has a root* t. *The* $f(t) = t^2 + t - 2m = 0$. *This implies* $t^2 + t = 2m$ *and* $t(t+1) = 2m$, *consequently* $m = \frac{t(t+1)}{2}$. ∎

Example 0.0.6 *Consider the quadratic function* $f(x) = x^2 + x - 30$. *Then* $f(x) = (x+6)(x-5) = 0$ *implies the roots of* $f(x)$ *are* $d = -6$ *or* $d = 5$.

Consequently, $m = \frac{d(d+1)}{2} = \frac{5(5+1)}{2} = \frac{-6(-6+1)}{2} = 15 = T_5$ *is a triangular number.*

Theorem 0.0.7 *Let* $f_i(x) = x^2 + x - 2T_i$ *and* $P(x) = \prod_{i=1}^{n} f_i(x)$ *where* T_i *and* R_i *are triangular numbers and roots to* $f_i(x)$ *respectively for each* $i \geq 1$. *Then*

1) $deg\ P(x) = 2n$, *and*
2) $\prod_{i=1}^{2n} R_i = (-1)^n 2^n \prod_{i=1}^{n} T_i$

Proof: *1) Given* $f_i(x) = x^2 + x - 2T_i$ *where* $T_i = \frac{i(i+1)}{2}$. *Then* $deg\, f_i(x) = 2$ *for each* $i \geq 1$.

For nonzero polynomial functions $f(x)$, $h(x)$, *and* $g(x)$ *such that* $f(x) = h(x)g(x)$,

$$deg\, f(x) = deg\, h(x) + deg\, g(x).$$

Hence $deg\big(P(x)\big) = deg(\prod_{i=1}^{n} f_i(x)) = \sum_{i=1}^{n} deg\, f_i(x) = \sum_{i=1}^{n} 2 = 2n$.

Consider $f_i(x) = x^2 + x - 2T_i = x^2 + x - i(i+1) = (x + (i+1))(x - i)$. *This implies* $f_i(x) = 0$ *if and only if* $\big(x + (i+1)\big)(x - i) = 0$ *if and only if* $x = i$ *or* $x = -(i+1)$. *Set* $R_i = i$ *or* $R_i = -(i+1)$. *Each quadratic polynomial function* $f_i(x)$ *has two distinct roots. This implies the product of all roots of the polynomial* $P(x)$,

$$\prod_{i=1}^{2n} R_i = \prod_{i=1}^{n} -(i+1)(i) = \prod_{i=1}^{n} -(i+1) \prod_{i=1}^{n} i = (-1)^n (n+1)!\,(n)!. \qquad (\circledast)$$

But $\prod_{i=1}^{n} T_i = \prod_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2^n} \prod_{i=1}^{n} i(i+1) = \frac{1}{2^n}(n!)(n+1)!$. *This implies*

$$2^n \prod_{i=1}^{n} T_i = n!\,(n+1)! \qquad (\circledast\circledast)$$

Combing $(\circledast)$ *and* $(\circledast\circledast)$ *we have* $\prod_{i=1}^{2n} R_i = (-1)^n 2^n \prod_{i=1}^{n} T_i$. ■

Define a sequence,

- $\{a_i\}_{i=1}^{\infty} = \left\{\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \dots\right\} = \left\{\frac{T_i}{T_{i+1}}\right\}_{i=1}^{\infty} = \{b_i\}_{i=1}^{\infty} \cup \{c_i\}_{i=1}^{\infty}$ *where*
- $\{b_i\}_{i=1}^{\infty} = \left\{\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9} \dots\right\} = \left\{\frac{2i-1}{2i+1}\right\}_{i=1}^{\infty} = \left\{\frac{T_{2i-1}}{T_{2i}}\right\}_{i=1}^{\infty} = \left\{\frac{f_i}{g_i}\right\}_{i=1}^{\infty}$ *and*
- $\{c_i\}_{i=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\right\} = \left\{\frac{T_{2i}}{T_{2i+1}}\right\}_{i=1}^{\infty} = \left\{\frac{h_i}{l_i}\right\}_{i=1}^{\infty}$ *and* $gcd(f_i, g_i) = gcd(h_i, l_i) = 1$, *[7]*

Set:

a) $\{d_i\}_{i=1}^{\infty} = \{1, 1, 3, 2, 5, 3, 7, 4, \dots\} = \bigcup_{i=1}^{\infty}\{2i-1,\ i\}$ *and*

b) $R_i = \{i(2i-1), i(2i+1) : i \geq 1\}$

c) For each $i \geq 0$, $\begin{cases} s_{2i} = 2i+1 \\ s_{2i+1} = i+1 \end{cases}$

Theorem 0.0.8 *Define for each* $i \geq 0$,

.

$$\begin{cases} s_{2i} = 2i + 1, \ (*) \\ s_{2i+1} = \ i + 1, (**) \end{cases}$$

Then $T_i = S_{i-1} * S_i$ *is a triangular number for each* $i \geq 1$.

Proof:

Case 1: *We first considered the case when* i *is even, i.e.,* $i = 2k$.

Then $T_{2k} = T_i = S_{i-1} * S_i$

$= S_{2k-1} * S_{2k} = S_{2(k-1)+1} * S_{2k},$ *because* $2k - 1 = 2(k-1) + 1$

$= ((k-1) + 1) * (2k+1)$ *by* $(*)$ *and* $(**)$

$= k * (2k+1) = \frac{2k*(2k+1)}{2} = \frac{i*(i+1)}{2}$

Therefore, $\boldsymbol{T_i} = \frac{\boldsymbol{i*(i+1)}}{\boldsymbol{2}}$, *and by (Theorem 0.0.1)* $T_i = S_{i-1} * S_i$ *is a triangular number.*

Case 2: *Now we considered the case when* i *is odd, i.e.,* $i = 2k + 1$.

This implies that $\square_{2k+1} = T_i = S_{i-1} * S_i$

$= S_{2k+1-1} * S_{2k+1}$

$= S_{2k} * S_{2k+1}, = (2k+1) * (k+1)$ *by* $(*)$ *and* $(**)$

$= \frac{(2k+1)*(2k+1+1)}{2} = \frac{i*(i+1)}{2},$

Therefore, $T_i = \frac{i*(i+1)}{2}$, *and by (Theorem 0.0.1),* $T_i = S_{i-1} * S_i$ *is a triangular number.* ■

Corollary 0.0.9 [8] **[A105658] off set {0}**

The set $F = \bigcup_{i=1}^{\infty}\{2i - 1, \ i\}$ *is the set of integers that satisfies the statement of (Theorem 0.0.10).*

$F = \{1,1,3,2,5,3,7,4,9,\dots\}$

Theorem 0.0.10 *Set* $R_i = \{h_i f_i = i(2i-1), \ h_i g_i = i(2i+1) : i \geq 1\}$.

Then $\bigcup_{i=1}^{n} R_i = \bigcup_{i=1}^{2\square} T_i$ (⊙⊙)

Proof: *Denote* $\eta_i = h_i f_i = i(2i-1)$ *and* $\mu_i = h_i g_i = i(2i+1)$ *for each* $i \geq 1$. *Then*
$$R_i = \{\eta_i, \mu_i | i \geq 1\}.$$

We set $F_n = \bigcup_{i=1}^{n} \eta_i$ *and* $G_n = \bigcup_{i=1}^{n} \mu_i$. *This implies* $\bigcup_{i=1}^{n} R_i = F_n \cup G_n$.
$= \bigcup_{i=1}^{n} \eta_i \cup \bigcup_{i=1}^{n} \mu_i$.

But $\eta_i = i(2i-1) = \frac{(2i-1)(2i)}{2} = T_{2i-1}, i \geq 1$ *and*

$$\mu_i = i(2i+1) = \frac{(2i)(2i+1)}{2} = T_{2i}, i \geq 1, \qquad (\odot\odot\odot)$$

are triangular numbers [7]. We use induction to prove the statement. We verify it is true for $n = 1$.

The left side of $(\odot\odot)$, $R_1 = F_1 \cup G_1 = \eta_1 \cup \mu_1 = \{T_1, T_2\} = \bigcup_{i=1}^{1} R_i$ *and the right side* $\bigcup_{i=1}^{2} T_i$, *are equal. Hence true for* $n = 1$. *Let* $k \in \mathbb{Z}^+$ *and suppose the statement in* $(\odot\odot)$ *is true for* $n = k$ *that is*

$$\bigcup_{i=1}^{k} R_i = \bigcup_{i=1}^{2k} T_i.$$

Now we show that it is true for $k = n+1$. *Thus*

$$\bigcup_{i=1}^{k+1} R_i = \bigcup_{i=1}^{k} R_i \cup \{R_{k+1}\} = \bigcup_{i=1}^{2k} T_i \cup \{R_{k+1}\} = \bigcup_{i=1}^{2k} T_i \cup \{\eta_{k+1}, \mu_{k+1}\}.$$

From $(\odot\odot\odot)$ $\eta_{k+1} = T_{2(k+1)-1} = T_{2k+1}$ *and* $\mu_{k+1} = T_{2(k+1)} = T_{2k+2}$. *This implies* $\bigcup_{i=1}^{2k} T_i \cup \{\eta_{k+1}, \mu_{k+1}\} = \bigcup_{i=1}^{2k} T_i \cup \{T_{2k+1}, T_{2k+2}\} = \bigcup_{i=1}^{2(k+1)} T_i$ *and* $\bigcup_{i=1}^{k+1} R_i = \bigcup_{i=1}^{2(k+1)} T_i$. *This implies the statement is true for* $k = n+1$.

Hence the statement in $(\odot\odot)$ *is true* $\forall\, k \geq 1$. ■

Definition 0.0.11

A finite or infinite expression of the form

$$a = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}} \qquad (*)$$

where the a_i *are real numbers with* $a_1, a_2, a_3, a_4, \ldots > 0$ *is called a continued fraction. The numbers* a_i *are called the* ***partial quotients*** *of the continued fraction.*

The continued fraction $(*)$ *is called* ***simple*** *if partial if the partial quotients* a_i *are all integers. It is*

called finite if it terminates, i.e., if it is of the form

$$a = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots + a_n}}} \qquad (**)$$

and infinite otherwise. [9, 10, 11]

Notation: *(Bracket notation for continued fractions). The continued fractions* $(*)$ *and* $(**)$ *are denoted by* $[a_1\,;\,a_2, a_3\,, a_4\,, \ldots\ ]$ *and* $[a_1\,;\,a_2, a_3\,, a_4\,, \ldots\ a_n]$ *respectively.*

Example 0.0.12

$$a = 1 + \frac{1}{1+[1\,;2\,,2,2,2,\ldots]} = 1 + \frac{1}{1+a}$$

Rearranging, we see a *must be a solution of* $x^2 = 2$ *, but since* a *is a positive (Indeed* $a > 1$*), we have* $a = \sqrt{2}$ *.*

Theorem 0.0.13 [12]

I. Continued fraction, $x = \cfrac{f_1}{f_1 + \cfrac{g_1}{g_1 + \cfrac{f_2}{f_2 + \cfrac{g_2}{g_2 + \cfrac{g_3}{\ldots}}}}} = \frac{1}{e-1}$ $(*)$

II. Let $a = \{f_1\ g_1,\ f_2\ ,\ g_2\,, f_3\,, g_3\,, \ldots\}$ *and* $b = \{\,h_1\,,\ l_1,\ h_2\ ,\ l_2\,,\ h_3\,, l_3\,, \ldots\}$*. The two set of numbers are generated by, GF,* $f(x) = \frac{1}{(1-x)(1-x^2)^2}$ *and* $g(x) = \frac{1+x+x^3+x^3}{(1-x^2)^2}$ *respectively.*

Some ODD and EVEN Triangular Numbers with Corresponding Subscripts [13]

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210

6	10	28	36	66	78	120	136	190	210	276	300	378	406
2*3	2*5	4*7	4*9	6*11	6*13	8*15	8*17	10*19	10*21	12*23	12*25	13*27	13*29
t_3	t_4	t_7	t_8	t_{11}	t_{12}	t_{15}	t_{16}	t_{19}	t_{20}	t_{23}	t_{24}	t_{27}	t_{28}

The table above shows even **triangular numbers** with their respective T subscripts (see shaded)

$$\begin{cases} t_{2i-2}, & i \text{ is even} \\ & \text{and} \\ t_{2i-1}, & i \text{ is odd} \end{cases} \Rightarrow \begin{cases} t_{4k-2}, & for\ i = 2k\ , k \in \mathbb{Z}^+ \\ & \text{and} \\ t_{4k-3}, & for\ i = 2k-1, k \in \mathbb{Z}^+ \end{cases}$$

1	3	15	21	45	55	91	105	153	171	231	253	325	351
1*1	1*3	3*5	3*7	5*9	5*11	7*13	7*15	9*17	9*19	11*21	11*23	13*25	13*27
t_1	t_2	t_5	t_6	t_9	t_{10}	t_{13}	t_{14}	t_{17}	t_{18}	t_{21}	t_{22}	t_{25}	t_{26}

The table above shows odd ***triangular numbers*** *with their respective T subscripts (see shaded).*

$$\begin{cases} t_{2i}\ , & i \text{ is even} \\ & \text{and} \\ t_{2i+1}, & i \text{ is odd} \end{cases} \Rightarrow \begin{cases} t_{4k}\ , & for\ i = 2k\ , k \in \mathbb{Z}^+ \\ & \text{and} \\ t_{4k-1}\ , & for\ i = 2k-1, k \in \mathbb{Z}^+ \end{cases}$$

Theorem 0.0.11: *Any two consecutive even triangular numbers have the form T_{4k} and T_{4k-1} for each $k \geq 1$, and $\sum_{i-1}^{n}(T_{4i}^2 - T_{4i-1}^2) = 64T_n^2$. Likewise any two consecutive odd triangular numbers are T_{4k-2} and T_{4k-3} for each $k \geq 1$ and*

$$\sum_{i-1}^{n}(T_{4i-2}^2 - T_{4i-3}^2) = 8T_{2n^2-1}^2 .$$

Proof: We prove the statement when

1) the triangular numbers have even parity,

$T_{4i} = \frac{4i(4i+1)}{2}$ and $T_{4i-1} = \frac{(4i-1)(4i)}{2}$ by Lemma (0.0.1). This implies,

$$T_{4i}^2 - T_{4i-1}^2 = \left(\frac{4i(4i+1)}{2}\right)^2 - \left(\frac{4i(4i-1)}{2}\right)^2 = \left(\frac{4i}{2}\right)^2((4i+1)^2 - (4i-1)^2) = 4i^2(4i) = (4i)^3$$

As $T_n = \frac{n(n+1)}{2}$ for each $n \geq 1$ and $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ we have

$$\sum_{i=1}^{n}(T_{4i}^2 - T_{4i-1}^2) = \sum_{i=1}^{n}(4i)^3 = 64\sum_{i-1}^{n} i^3 = 64\frac{n^2(n+1)^2}{4} = 64\left(\frac{n(n+1)}{2}\right)^2 = 64T_n^2 .$$

2) the triangular numbers have odd parity,

$T_{4i-2} = \frac{(4i-2)(4i-1)}{2}$ and $T_{4i-3} = \frac{(4i-3)(4i-2)}{2}$, (Lemma 0.0.1) . This implies

$$T_{4i-2}^2 - T_{4i-3}^2 = \left(\frac{(4i-2)(4i-1)}{2}\right)^2 - \left(\frac{(4i-3)(4i-2)}{2}\right)^2 = \left(\frac{(4i-2)}{2}\right)^2((4i-1)^2 - (4i-3)^2)$$

$$= (2i-1)^2(16i-2) = 8(2i-1)^3 .$$

Therefore,

$\sum_{i-1}^{n}(T_{4i-2}^2 - T_{4i-3}^2) = \sum_{i-1}^{n} 8(2i-1)^3 = 8\sum_{i-1}^{n}(2i-1)^3$. Let $2i - 1 = k$.

Hence $\sum_{i-1}^{n}(T_{4i-2}^2 - T_{4i-3}^2) = 8\sum_{k-1}^{2n-1} k^3 = 8T_{2n^2-1}^2$. ⊡

ACKNOWLEDGEMENTS

The authors thank the Savannah State University Re- Search Council for financial support through the 2016 NSF-Magic-Stem plus Program Research Grant.

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