CURVILINEAR CO-ORDINATES AND EXPRESSION OF MOTION EQUATIONS FROM VECTOR FORM TO CYLINDRICAL CO-ORDINATES

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ABSTRACT
In this paper, we consider at the equations of motion in vector form and convert them to cylindrical coordinates. We also consider the continuity and momentum properties of these equations.

INTRODUCTION

Cartesian coordinates
In Cartesian coordinates, the Navier-Stokes and the continuity equations are given by;

Continuity equation
The general form of equation of conservation of mass is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{1}$$

In equation above, u, v and w are velocities in x, y and z- directions and ρ is the density.

The above equation is valid for steady and unsteady, compressible and incompressible fluid. In vector form, equation can be written in vector form as;

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{q}) = 0} \tag{2}$$

Equation (1) is the first form of continuity equation

Here, $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\boldsymbol{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$

The continuity equation can also be written in another form, we use product rule on the divergent term in equation (2) to get,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{q}) = \frac{\partial \rho}{\partial t} + \boldsymbol{q} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{q} = 0 \qquad (3)$$

In equation (3) above, the terms, $\frac{\partial \rho}{\partial t} + \boldsymbol{q} \cdot \nabla \rho$ can be replaced by the material derivative, $\frac{D\rho}{Dt}$,

therefore equation (2) will become,

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla . \boldsymbol{q} = 0} \qquad (4)$$

Equation four is the second form of continuity equation

There are two special cases for the continuity equation (2)

1. For Steady flow, the equation does not depend on time, therefore equations (1) and (2) becomes $\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

   (5)

   Or in vector form

   $$\nabla . \rho\, \boldsymbol{q} = 0 \qquad (6)$$

   this follows since by definition, ρ is not a function of time for steady flow, but could be a function of position.

2. For incompressible fluids the density, ρ is constant throughout the flow field so that the equations (1) and (2) become;

   $$\nabla \cdot \boldsymbol{q} = 0 \qquad (7)$$

   Or

   $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (8)$$

The above equation is a special form of equation (5) when density is not a function of position and it applies to both steady and unsteady flow of incompressible fluids; this is the equation we will use in this project since we will assume that blood is incompressible as stated in the assumptions.

MAIN RESULTS

The Navier stokes equations

Navier-Stokes equations are the equations of conservation of linear momentum. The general form of the equations for incompressible flow of Newtonian (constant viscosity) fluid is given by;

$$\frac{\partial \boldsymbol{q}}{\partial t} = -(\boldsymbol{q}.\boldsymbol{\nabla})\boldsymbol{q} + \upsilon\nabla^2\boldsymbol{q} - \frac{1}{\rho}\nabla P + \boldsymbol{g} + \boldsymbol{f} \qquad \textbf{(9)}$$

$\upsilon$is kinetic viscosity (constant) and is given by $\upsilon = \frac{\mu}{\rho}$, ρ is density (constant), $\boldsymbol{P}$ is pressure and $\boldsymbol{g}$is the gravitational force.

In the equation (9) above,

- $\frac{\partial \boldsymbol{q}}{\partial t}$ – Acceleration term
- $(\boldsymbol{q}.\boldsymbol{\nabla})\boldsymbol{q}$ **–** is the advection term; the force exerted on the particles of the fluid by other particles of the fluid surrounding it
- $\upsilon\nabla^2\boldsymbol{q}$ **–** velocity diffusion terms; describes how the fluid motion is damped, highly viscous fluid e.g. honey stick together while low viscous fluid flow freely, e.g. air
- $\boldsymbol{\nabla}P$ - pressure term, fluids flow in the direction of largest change in pressure

From equation (2), $\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and$\boldsymbol{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$**.** Replacing this in equation (9) we obtain

$$\frac{\partial}{\partial t}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = -(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w).\left(\mathbf{i}\frac{\partial}{\partial \mathrm{x}} + \mathbf{j}\frac{\partial}{\partial \mathrm{y}} + \mathbf{k}\frac{\partial}{\partial \mathrm{z}}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) +$$

$$\upsilon\nabla^2(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) - \frac{1}{\rho}\left(\mathbf{i}\frac{\partial}{\partial \mathrm{x}} + \mathbf{j}\frac{\partial}{\partial \mathrm{y}} + \mathbf{k}\frac{\partial}{\partial \mathrm{z}}\right)P + \rho(\mathbf{i}g_x + \mathbf{j}g_x + \mathbf{k}g_x) + (\mathbf{i}f_x + \mathbf{j}f_x + \mathbf{k}f_x)$$

(10)

In equation (10) the laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial \mathrm{x}^2} + \frac{\partial^2}{\partial \mathrm{y}^2} + \frac{\partial^2}{\partial \mathrm{z}^2}$$

Replacing the laplacian above in equation (10), we obtain

$$\frac{\partial}{\partial t}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = \left(\mathrm{u}\frac{\partial}{\partial \mathrm{x}} + v\frac{\partial}{\partial \mathrm{y}} + w\frac{\partial}{\partial \mathrm{z}}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) + \upsilon\left(\frac{\partial^2}{\partial \mathrm{x}^2} + \frac{\partial^2}{\partial \mathrm{y}^2} + \frac{\partial^2}{\partial \mathrm{z}^2}\right)$$

$$(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) - \frac{1}{\rho}\left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right)P + (\boldsymbol{i}g_x + \boldsymbol{j}g_x + \boldsymbol{k}g_x) + (\boldsymbol{i}f_x + \boldsymbol{j}f_x + \boldsymbol{k}f_x) \tag{11}$$

Collecting coefficients of, $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ together, this leads to equations in x, y and z directions respectively as follows;

$$\frac{\partial u}{\partial t} = \left(\mathrm{u}\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u + \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + f_x \tag{12}$$

$$\frac{\partial v}{\partial t} = \left(\mathrm{u}\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v + \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + f_y$$

$$\frac{\partial w}{\partial t} = \left(\mathrm{u}\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w + \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)w + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + f_z$$

The above three equations are the Navier-Stokes equations in x, y and z components. In this project we will neglect the body forces. Therefore dropping the body forces and rearranging the equations we obtain;

x-component

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{13}$$

y-component

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{14}$$

z-component

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{15}$$

Here, υ is the coefficient of viscosity, ρ is the density of the fluid and $\mathbf{g}$ is the gravitational force

Equations of motion in cylindrical coordinates

In cylindrical coordinates, the coordinates r is the radial distance from the z axis, θ is the angle measured from a line parallel to the x- axis, z is the coordinates along the z- axis. The velocity components are the radial velocity u, the tangential velocity v, and the axial velocity w. Thus the velocity at some arbitrary point p can be expressed as

$$\mathbf{q} = u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_z\mathbf{e_z} \tag{16}$$

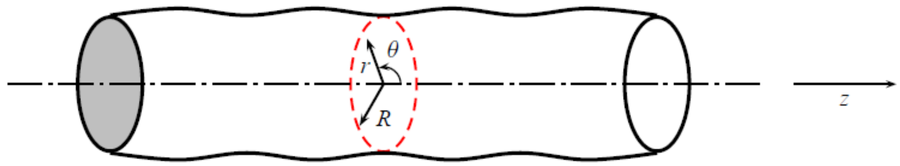

Figure 1: Shape of an artery

Cartesian coordinates can be expressed into cylindrical coordinates using the relations,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad z = z \tag{17}$$

The above relations implies that

$$\begin{aligned} x &= x(r,\theta) \\ y &= y(r,\theta) \\ z &= z \end{aligned} \tag{18}$$

I.e. Cartesian coordinates can be expressed in terms of cylindrical coordinates, also

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad r = \sqrt{x^2 + y^2}, \qquad z = z \tag{19}$$

This relations also implies that

$$\theta = \theta(x, y), \qquad r = r(x, y), \qquad z = z \tag{20}$$

Therefore equations (20) and (18) show the relationship between Cartesian coordinates and cylindrical coordinates. In general form, the relationship between Cartesian coordinates and any other coordinate system can be represented by,

$$\begin{aligned} x &= x(u_1,u_2,u_3) & \qquad u_1 &= u_1(x,y,z) \\ y &= y(u_1,u_2,u_3) & \qquad u_2 &= u_2(x,y,z) \\ z &= z(u_1,u_2,u_3) & \qquad u_3 &= u_3(x,y,z) \end{aligned}$$

In the above equation, u_1, u_2, u_3 are the curvilinear coordinates, they can be cylindrical coordinates or spherical coordinates. In cylindrical coordinates;

$$u_1 = u_r = r,\ u_2 = u_\theta = \theta \text{ and } \mathrm{u}_3 = u_z = z \tag{21}$$

To convert Cartesian coordinates to cylindrical coordinates, we first convert them into curvilinear coordinates before to cylindrical coordinates.

In this section we will transform the continuity and momentum equations from Cartesian to cylindrical coordinates. We will start by converting the continuity equation, and then followed by the momentum equations.

Continuity equation

The continuity equation in vector form as shown in (7) is given by;

$\nabla \cdot \boldsymbol{q} = 0$

In order to understand how to convert the equation in curvilinear coordinates, we first need to first know several parameters which we are going to use.

Unit vectors (in curvilinear coordinates)

In curvilinear coordinates u_1, u_2 and u_3 the unit vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are given by

$$\mathbf{e}_i = \frac{\partial \mathbf{r}/\partial u_i}{\left|\partial \mathbf{r}/\partial u_i\right|} = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u_i} \tag{22}$$

Where, $i = 1, 2, 3$

$\mathbf{r}$ is a position vector of any point in Cartesian coordinate system and is given by;

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{23}$$

$\partial \mathbf{r}/\partial u_i$ Is a vector in the direction of the tangent to the u_i - curve. In cylindrical coordinates, the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are given as $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$.

Scale factors

We will take the curvilinear coordinates u_1, u_2 and u_3 to be orthogonal. From equation (22), the scale factors are h_i where $i = 1, 2, 3$ and are given by;

$$h_i = \left|\frac{\partial \mathbf{r}}{\partial u_i}\right| \tag{24}$$

In equation (24) above,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{25}$$

Replacing equation (25) into equation (24) we obtain the equation,

$$h_i = \left|\frac{\partial \tilde{r}}{\partial u_i}\right| = \left|\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right| \tag{26}$$

$i = 1, 2, 3$

From equation (26) above, we get,

$$h_i^2 = \left(\frac{\partial \tilde{r}}{\partial u_i}\right)\left(\frac{\partial \tilde{r}}{\partial u_i}\right) = \left(\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right) \bullet \left(\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right)$$
$$= \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \qquad (27)$$

Replacing equation (17) into (27), we obtain

$$h_i^2 = \left(\frac{\partial}{\partial u_i} r\cos\theta\right)^2 + \left(\frac{\partial}{\partial u_i} r\sin\theta\right)^2 + \left(\frac{\partial}{\partial u_i} z\right)^2 \qquad (28)$$

$i = 1,$ 2 and 3

In cylindrical coordinates the scale factors; h_1, h_2 and h_2 and the coordinates u_1, u_2 and u_3 are given by;

$$h_r = h_1 \quad h_\theta = h_2 \text{ and } \mathrm{h}_z = h_3$$
$$u_1 = u_r = r, \quad u_2 = u_\theta = \theta, \quad u_3 = u_z = z \qquad (29)$$

Now replacing (29) into (28) above we get

$$h_r^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial}{\partial r}(r\cos\theta)\right)^2 + \left(\frac{\partial}{\partial r}(r\sin\theta)\right)^2 + \left(\frac{\partial}{\partial r} z\right)^2 \qquad (30)$$
$$= \cos^2\theta + \sin^2\theta = 1$$

$$h_\theta^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial}{\partial \theta}(r\cos\theta)\right)^2 + \left(\frac{\partial}{\partial \theta}(r\sin\theta)\right)^2 + \left(\frac{\partial}{\partial \theta} z\right)^2 \qquad (31)$$
$$= r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$$h_z^2 = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = \left(\frac{\partial}{\partial z}(r\cos\theta)\right)^2 + \left(\frac{\partial}{\partial z}(r\sin\theta)\right)^2 + \left(\frac{\partial}{\partial z} z\right)^2 = 1 \qquad (32)$$

From the equations; (30), (31) and (32), the scale factors in cylindrical coordinates are;

$$h_r = 1, \quad h_\theta = r, \quad h_z = 1 \qquad (33)$$

Now after the above explanations, we now express the continuity equation $\nabla\cdot\boldsymbol{q} = 0$ in cylindrical coordinates. We first express the divergence, $\nabla\cdot\boldsymbol{q}$ in curvilinear coordinates. Before we do this, we first find the value of Del operator, ∇in curvilinear coordinates. To do this, we first find the value of$\nabla\emptyset$, where $\emptyset$ is any scalar function. In curvilinear coordinates, it will be written as

$$\nabla\emptyset = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3 \quad (34)$$

$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$, are unit vectors along u_1, u_2, u_3 curves.

Let $\boldsymbol{r}$be a position vector of a point $\boldsymbol{P}$in Cartesian coordinates. Then$\boldsymbol{r}$ from (25) is given by

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Using the relations (17) the equation above becomes

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k} \quad (35)$$

We need to express $d\emptyset$ in two different ways and compare the coefficients of du_1, du_2 and du_3 to obtain the values of, f_1, f_2, and f_3 in equation (34).

In the first expression,

$$\begin{aligned} d\phi &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right) \bullet (\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz) \\ &= \nabla\phi \bullet dr \end{aligned} \quad (36)$$

Then using equation

$\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$

$$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial u_1}du_1 + \frac{\partial\mathbf{r}}{\partial u_2}du_2 + \frac{\partial\mathbf{r}}{\partial u_2}du_3 \quad (37)$$

From (22), equation (37) can be written as

$$d\mathbf{r} = h_1du_1\mathbf{e}_1 + h_2du_2\mathbf{e}_2 + h_3du_3\mathbf{e}_3 \quad (38)$$

We therefore obtain

$$\begin{aligned} d\phi &= \nabla\phi.dr = (f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3).(h_1\mathbf{e}_1du_1 + h_2\mathbf{e}_2du_2 + h_3\mathbf{e}_3du_3) \\ &= f_1h_1du_1 + f_2h_2du_2 + h_3f_3du_3 \end{aligned} \quad (39)$$

$d\phi$ can also be expressed as,

$$d\phi = \frac{\partial\phi}{\partial u_1}du_1 + \frac{\partial\phi}{\partial u_2}du_2 + \frac{\partial\phi}{\partial u_2}du_3 \quad (40)$$

Comparing equations (39) and equations (40), get the following,

$$f_1 = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1} \qquad f_2 = \frac{1}{h_2}\frac{\partial \phi}{\partial u_2} \qquad f_3 = \frac{1}{h_3}\frac{\partial \phi}{\partial u_3} \tag{41}$$

Replacing (41) in (34) we obtain,

$$\boldsymbol{\nabla}\phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\mathbf{e_3} \tag{42}$$

Therefore, from equation (42)

$$\boldsymbol{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial}{\partial u_3}\mathbf{e_3} \tag{43}$$

In cylindrical coordinates, equation (43) can be written as

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e_\theta} + \frac{\partial}{\partial z}\mathbf{e_z} \tag{44}$$

We will then proceed to find the value of $\boldsymbol{\nabla} \cdot \boldsymbol{q}$in curvilinear coordinates. In curvilinear coordinates, we will take$\boldsymbol{q}$to be equal to

$$\boldsymbol{q} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3}$$

$$\nabla \cdot \mathbf{q} = \nabla \bullet (u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}) = \nabla \cdot (u_1\mathbf{e_1}) + \nabla \cdot (u_2\mathbf{e_2}) + \nabla \cdot (u_3\mathbf{e_3}) \tag{45}$$

From, equation (42) above,

$$\nabla u_1 = \frac{1}{h_1}\frac{\partial u_1}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial u_1}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial u_1}{\partial u_3}\mathbf{e}_3 = \frac{\mathbf{e_1}}{h_1} \tag{46}$$

$$\nabla u_2 = \frac{1}{h_1}\frac{\partial u_2}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial u_2}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial u_3}{\partial u_3}\mathbf{e_3} = \frac{\mathbf{e_2}}{h_2} \tag{47}$$

$$\nabla u_3 = \frac{1}{h_1}\frac{\partial u_3}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial u_3}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial u_3}{\partial u_3}\mathbf{e_3} = \frac{\mathbf{e_3}}{h_3} \tag{48}$$

We then deal with each term on the right hand-side of equation (45), but first we derive certain relations. From equations (46), (47) and (48), we obtain;

$$\frac{\mathbf{e_1}}{h_2 h_3} = \frac{\mathbf{e_2} \times \mathbf{e_3}}{h_2 h_3} = \nabla u_2 \times \nabla u_3 \quad \Rightarrow \mathbf{e_1} = h_2 h_3 \nabla u_2 \times \nabla u_3 \tag{49}$$

$$\frac{\mathbf{e_2}}{h_1 h_3} = \frac{\mathbf{e_1} \times \mathbf{e_3}}{h_1 h_3} = \nabla u_1 \times \nabla u_3 \quad \Rightarrow \mathbf{e_2} = h_1 h_3 \nabla u_1 \times \nabla u_3 \tag{50}$$

$$\frac{\mathbf{e_3}}{h_1 h_2} = \frac{\mathbf{e_1} \times \mathbf{e_2}}{h_1 h_2} = \nabla u_1 \times \nabla u_2 \quad \Rightarrow \mathbf{e_3} = h_1 h_2 \nabla u_1 \times \nabla u_2 \tag{51}$$

We will then express all the terms on the left hand-side in equation (45) in curvilinear coordinates, we will begin with the term, $\nabla \cdot (u\mathbf{e_1})$

$$\begin{aligned} \nabla \cdot (u_1 \mathbf{e_1}) &= \nabla \cdot \left(h_2 h_3 u_1 \nabla u_2 \times \nabla u_3\right) \\ &= h_2 h_3 u_1 \nabla \cdot \left(\nabla u_2 \times \nabla u_3\right) + \left(\nabla u_2 \times \nabla u_3\right).\nabla\left(h_2 h_3 u_1\right) \end{aligned} \tag{52}$$

But in equation (52) above,

$$\nabla \cdot \left(\nabla u_2 \times \nabla u_3\right) = \nabla u_3 \cdot \left(\nabla \times \nabla u_2\right) - \nabla u_2 \cdot \left(\nabla \times \nabla u_3\right) = 0$$

Hence,

$$\nabla \cdot (u_1 \mathbf{e_1}) = \left(\nabla u_2 \times \nabla u_3\right).\nabla\left(h_2 h_3 u_1\right) \tag{53}$$

From equations, (47 – 49), equation (53) becomes

$$\nabla \cdot (u_1 \mathbf{e_1}) = \frac{\mathbf{e_1}}{h_2 h_3}.\nabla\left(h_2 h_3 u_1\right) \tag{54}$$

Using equation (42), we can express $\nabla(h_2 h_3 u_1)$ in the above equation as

$$\nabla\left(h_2 h_3 u_1\right) = \frac{1}{h_1}\frac{\partial\left(h_2 h_3 u_1\right)}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial\left(h_2 h_3 u_1\right)}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial\left(h_2 h_3 u_1\right)}{\partial u_3}\mathbf{e_3} \tag{55}$$

Now replacing equation (55) into (54) we obtain

$$\nabla \cdot (u_1 \mathbf{e_1}) = \frac{\mathbf{e_1}}{h_2 h_3} \cdot \left(\frac{1}{h_1}\frac{\partial}{\partial u_1}\left(h_2 h_3 u_1\right)\mathbf{e_1} + \frac{1}{h_2}\frac{\partial}{\partial u_2}\left(h_2 h_3 u_1\right)\mathbf{e_2} + \frac{1}{h_3}\frac{\partial}{\partial u_3}\left(h_2 h_3 u_1\right)\mathbf{e_3}\right) \tag{56}$$

Simplifying equation (56), we obtain,

$$\nabla \cdot (u_1 \mathbf{e_1}) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}\left(h_2 h_3 u_1\right) \tag{57}$$

In a similar way the remaining two terms in equation (42) can be found to be

$$\nabla \cdot (u_2 \mathbf{e_2}) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}\left(h_1 h_3 u_2\right) \tag{58}$$

$$\nabla \cdot (u_3 \mathbf{e_3}) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}\left(h_1 h_2 u_3\right) \tag{59}$$

Replacing equations (57 – 59) in (45) we obtain

$$\nabla \cdot \mathbf{q} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}\left(h_2 h_3 u_1\right) + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}\left(h_1 h_3 u_1\right) + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}\left(h_1 h_2 u_1\right) \tag{60}$$

Using equations (33) we can now express equation (60) in cylindrical coordinates as

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{\partial\left(r u_r\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(u_\theta\right)}{\partial \theta} + \frac{\partial\left(u_z\right)}{\partial z} \tag{61}$$

Momentum equation

The momentum equation in vector form as shown in equation (9)

$$\rho\left(\frac{\partial q}{\partial t} + \left(\mathbf{q}.\nabla\right)\mathbf{q}\right) = -\nabla p + p\mathbf{g} + \mu \nabla^2 \mathbf{q}$$

We want to express the above equation in cylindrical coordinates. Since we have already calculated the value of the grad, ∇ we only need to find the value of the Laplacian, ∇^2. To do this, we first find the value of $\boldsymbol{\nabla^2 f}$, where $\boldsymbol{f}$ is any scalar function and then and then finally find the laplacian.

$$\nabla^2 f = \nabla \cdot \left(\nabla f\right) \tag{62}$$

We then let

$$\boldsymbol{F = \nabla f}, \text{ where } \boldsymbol{F = F_1 e_1 + F_2 e_2 + F_3 e_3} \tag{63}$$

From equation (42),

$$F = \nabla f = \frac{\mathbf{e_1}}{h_1}\frac{\partial f}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\mathbf{e_3}}{h_3}\frac{\partial}{\partial u_3} \tag{64}$$

Comparing coefficients of $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$, in equations (63), (64), we obtain;

$$F_1 = \frac{1}{h_1}\frac{\partial f}{\partial u_1},\ F_2 = \frac{1}{h_2}\frac{\partial f}{\partial u_2},\ F_3 = \frac{1}{h_3}\frac{\partial f}{\partial u_3} \tag{65}$$

From equation (60), equation (62) can be written as,

$$\begin{aligned}\nabla^2 f &= \nabla \cdot (\nabla f) = \nabla \cdot \mathbf{F} = \nabla \cdot (F_1 \mathbf{e_1} + F_2 \mathbf{e_2} + F_3 \mathbf{e_3}) \\ &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(h_2 h_3 F_1\right) + \frac{\partial}{\partial u_2}\left(h_1 h_3 F_2\right) + \frac{\partial}{\partial u_3}\left(h_1 h_2 F_3\right)\right]\end{aligned} \tag{66}$$

Equation (66) can be simplified to

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right] \tag{67}$$

Hence the laplacian in curvilinear coordinates is given by;

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right] \tag{68}$$

In cylindrical coordinates, equation (68) becomes

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2} \tag{69}$$

The gravitational force, $\boldsymbol{g}$ in cylindrical coordinates can be written as,

$$\mathbf{g} = g_r\mathbf{e_r} + g_\theta \mathbf{e_\theta} + g_z \mathbf{e_z} \tag{70}$$

Substituting, ∇, ∇^2 and $\mathbf{g}$ from equations; (42), (69) and (70) respectively, in equation (60),

$$\rho\left(\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\mathbf{q}\right) = -\nabla p + p\mathbf{g} + \mu\nabla^2\mathbf{q}$$

We obtain;

$$\rho\left\{\frac{\partial}{\partial t}(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})+\left[(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z}).\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+\frac{\partial}{\partial z}\mathbf{e_z}\right)\square(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})\right]\right\} =$$
$$-\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+\frac{\partial}{\partial z}\mathbf{e_z}\right)p + p(g_r\mathbf{e_r}+g_\theta\mathbf{e_\theta}+g_z\mathbf{e_z})+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right](u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z}) \tag{71}$$

On simplifying equation (71), we

obtain

$$\rho\frac{\partial}{\partial t}u_r\mathbf{e_r}+\rho\frac{\partial}{\partial t}u_\theta\mathbf{e_\theta}+\rho\frac{\partial}{\partial t}u_z\mathbf{e_z}+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)\square(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})$$
$$=-\left(\frac{\partial p}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial p}{\partial\theta}\mathbf{e_\theta}+\frac{\partial p}{\partial z}\mathbf{e_z}\right)+(pg_r\mathbf{e_r}+pg_\theta\mathbf{e_\theta}+pg_z\mathbf{e_z})$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right](u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})$$

$$\rho\frac{\partial}{\partial t}u_r\mathbf{e_r}+\rho\frac{\partial}{\partial t}u_\theta\mathbf{e_\theta}+\rho\frac{\partial}{\partial t}u_z\mathbf{e_z}+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)u_r\mathbf{e_r}+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)u_\theta\mathbf{e_\theta}$$
$$+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)u_z\mathbf{e_z}=-\left(\frac{\partial p}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial p}{\partial\theta}\mathbf{e_\theta}+\frac{\partial p}{\partial z}\mathbf{e_z}\right)+\left(pg_r\mathbf{e_r}+pg_\theta\mathbf{e_\theta}+pg_z\mathbf{e_z}\right)+$$
$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]u_r\mathbf{e_r}+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]u_\theta\mathbf{e_\theta}+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]u_z\mathbf{e_z} \quad (73)$$

Now collecting terms that contain $\mathbf{e_r}$, we obtain;

$$\rho\frac{\partial}{\partial t}u_r+\rho\left(u_r\frac{\partial}{\partial r}+u_\theta\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)u_r=-\frac{\partial p}{\partial r}+pg_r+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)u_r \quad (74)$$

While those that contain $\mathbf{e_\theta}$, are

$$\rho\frac{\partial}{\partial t}u_\theta+\rho\left(u_r\frac{\partial}{\partial r}+u_\theta\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)u_\theta=-\frac{\partial p}{\partial\theta}+pg_\theta+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)u_\theta \quad (75)$$

And finally those containing e_z are

$$\rho\frac{\partial}{\partial t}u_z+\rho\left(u_r\frac{\partial}{\partial r}+u_\theta\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)u_z=-\frac{\partial p}{\partial z}+pg_z+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)u_z \quad (76)$$

Therefore, equations; (74), (75) and (76) are the Navier – Stokes equation in cylindrical coordinates. On simplifying the three equations we obtain

r-component

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial r}+\rho q_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}\right] \quad (77)$$

θ- component

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\rho q_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right] \quad (78)$$

z- component

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\rho q_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \tag{79}$$

CONCLUSION

In this paper we have outlined the expression of motion equations from vector form to cylindrical coordinates.

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