Some Celebrity Theorems of the Mulatu Numbers

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Abstract

The Mulatu numbers are sequences of numbers of the form 4, 1, 5, 6, 11, 17, 28, 45, ... The numbers have wonderful and amazing properties and patterns. In mathematical terms, it is defined by the following recurrence relation:

$$M_n = \begin{cases} 4 & if\ n = 0; \\ 1 & if\ n = 1; \\ M_{n-1} + M_{n-2} & if\ n > 1. \end{cases}$$

The first number of the sequence is 4, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers of the sequence itself. That is, after two starting values, each number is the sum of the two preceding numbers. In this paper, we investigate the effect of the Core Theorem given below in providing different new proofs for some important results which are already known.

1. Introductions and Background

The Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, an Ethiopian Mathematician and Professor of Mathematics at Savannah State University, Savannah, Georgia, USA. The numbers are closely related to both Fibonacci and Lucas Numbers in its properties and patterns. Below we give the First 20 Mulatu, Fibonacci and Lucas numbers.

First 20 Mulatu, Fibonacci and Lucas Numbers (Tables 1 & 2).

Table 1

n :	0	1	2	3	4	5	6	7	8	9	10	11
M_n :	4	1	5	6	11	17	28	45	73	118	191	309
F_n :	0	1	1	2	3	5	8	13	21	34	55	89
L_n :	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n :	12	13	14	15	16	17	18	19	20
M_n :	500	809	1309	2118	3427	5545	8972	14517	23489
F_n :	144	233	377	610	987	1597	3584	4181	6765
L_n :	322	521	843	1364	2207	3571	5778	9349	15127

Remark 1 : Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers, Fibonacci numbers, and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $F_{2n} = F_n L_n$

(4) $L_n = F_n + 2F_{n-1}$

(5) $F_n = \frac{L_{n+1} + L_{n-1}}{5}$

(6) $L_{n+1} = L_n + L_{n-1}$

(7) $F_{n+k} = F_{n-1}F_k + F_n F_{k+1}$

(8) $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$

(9) $L_{n+m} = \frac{5F_nF_m + L_nL_m}{2}$

(10) $M_{n+k} = F_{n-1}M_k + M_nF_{k+1}$

2. The Main Results

Theorem 1. $L_n = \frac{M_n + F_n}{2}$

Proof: We use indication on n.

1. When n=1, the formula is true as $L_1 = \frac{M_1+F_1}{2} = \frac{1+1}{2} = 1$
2. Assume the formula is true for $n = 1,2,3 \dots k-1, k$
3. Verify the formula for $n = k+1$.

4. $L_{k+1} = L_k + L_{k-1}$

$$= \frac{M_k+F_k}{2} + \frac{M_{k-1}+F_{k-1}}{2}$$
$$= \frac{M_k+M_{k-1}+F_k+F_{k-1}}{2}$$
$$= \frac{M_{k+1}+F_{k+1}}{2}$$

Corollary 1. $M_n = L_n + 2F_{n-1}$

Proof. $L_n = \frac{M_n+F_n}{2}$

$$\Rightarrow M_n = 2L_n - F_n$$
$$\Rightarrow M_n = L_n + L_n - F_n$$
$$\Rightarrow M_n = L_n + F_{n+1} + F_{n-1} - F_n$$
$$= L_n + F_n + F_{n-1} + F_{n-1} - F_n$$
$$= L_n + 2F_{n-1}$$

Corollary 2 $M_n = F_n + 4F_{n-1}$

Proof. $M_n = L_n + 2F_{n-1}$ **(by Corollary 1)**

$$= F_{n+1} +\backslash+2F_{n-1}$$
$$= F_n + F_{n-1} + F_{n-1} + 2F_{n-1}$$
$$= F_n + 4F_{n-1}$$

Corollary 3. $M_n = \frac{7L_n+2L_{n-2}}{5}$

Proof. $M_n = 2L_n - F_n$

$$\Rightarrow 5M_n = 10L_n - 5F_n$$
$$= 7L_n + 3L_n - 5F_n$$
$$= 7L_n + 2L_n + L_n - 5F_n$$
$$= 7L_n + 2(L_{n-1} + L_{n-2}) + L_n - 5F_n$$
$$= 7L_n + 2L_{n-1} + 2L_{n-2} + L_n - 5F_n$$
$$= 7L_n + 2L_{n-2} + 2F_n + 2F_{n-2} + F_n + 2F_{n-1} - 5F_n$$
$$= 7L_n + 2L_{n-2} + 3F_n + 2(F_{n-1} + F_{n-2}) - 5F_n$$
$$= 7L_n + 2L_{n-2} + 3F_n + 2F_n - 5F_n$$
$$= 7L_n + 2L_{n-2}$$
$$\Rightarrow M_n = \frac{7L_n+2L_{n-2}}{5}$$

Corollary 4. $M_n = F_{n-3} + F_{n-1} + F_{n-2}$

Proof. $M_n = L_n + L_n - F_n$

$$\Rightarrow M_n = F_{n+1} + F_{n-1} + F_{n+1} + F_{n-1} - F_n$$
$$= F_{n+1} + F_{n-1} + F_n + F_{n-1} + F_{n-1} = (F_{n-1} - F_{n-2})$$
$$= F_{n+1} + F_{n-1} + F_n + F_{n-1} + F_{n-2}$$
$$= F_{n+1} + F_n + F_{n-1} + F_{n-3}$$
$$= F_{n+2} + F_{n-1} + F_{n-3}$$
$$= F_{n-3} + F_{n-1} + F_{n-2}$$

Corollary 5. $M_{2n} = M_n L_n + 4(-1)^{n+1}$

Proof. $M_n = 2L_n - F_n$

$$\Rightarrow M_{2n} = 2L_{2n} - F_{2n}$$
$$= 5F_n^2 + L_n^2 - F_n L_n \text{ (by 3 and 9 above)}$$
$$= 5F_n^2 + L_n(L_n - F_n)$$
$$= 5F_n^2 + L_n(M_n - L_n)$$
$$= 5F_n^2 + M_n L_n - L_n^2$$
$$= M_n L_n + 5F_n^2 - L_n^2$$
$$= M_n L_n + 4(-1)^{n+1} \text{ (by 8 above)}$$

Corollary 6. $L_{2n} + 2F_{2n-1} = M_{2n}$

Proof. $L_{2n} + 2F_{2n-1}$

$$= 2L_{2n} + 2F_{2n-1} - L_{2n}$$
$$= 2L_{2n} + 2F_{2n-1} - (F_{2n} + 2F_{2n-1}) \text{ (by 4 above)}$$
$$= 2L_{2n} + 2F_{2n-1} - F_{2n} - 2F_{2n-1}$$
$$= 2L_{2n} - F_{2n}$$
$$= M_{2n}$$

Theorem 2. **Some Divisibility Properties of M.**

(a) If M_n is divisible by 2, then $M^2{}_{n+1} - M^2{}_{n-1}$ is divisible by 4

(b) If M_n is divisible by 3, then $M^3{}_{n+1} - M^3{}_{n-1}$ is divisible by 9.

Proof: Note that: Using $M_{n+1} = (M_n + M_{n-1})$, we have:

(a) $M^2{}_{n+1} - M^2{}_{n-1}$
$$= (M_{n+1} - M_{n-1})(M_{n+1} + M_{n-1}) = M_n\,(M_n + M_{n-1} + M_{n-1}) = M^2{}_n + 2M_n M_{n-1}.$$

Now it is easy to see that if M_n is divisible by 2, then $M^2{}_{n+1} - M^2{}_{n-1}$ is divisible by 4

(b) $M^3{}_{n+1} - M^3{}_{n-1} = (M_{n+1} - M_{n-1})(M^2{}_{n+1} + M_n M_{n-1} + M^2{}_{n-1})$

$$= M_n\,(M^2{}_{n+1} + M_{n+1}M_{n-1} + M^2{}_{n-1})$$

$$= M_n\left((M_n + M_{n-1})^2 + M_{n-1}(M_n + M_{n-1}) + M^2{}_{n-1}\right)$$

$$= M_n\left(M^2{}_n + 3M_n M_{n-1} + 3M^2{}_{n-1}\right)$$

$$= M^3{}_n + 3M^2{}_n M_{n-1} + 3M_n M^3{}_{n-1}$$

Hence M_n is divisible by 3 $\Rightarrow$ $M^3{}_{n+1}$ - $M^3{}_{n-1}$ is divisible by 9.

Theorem 3. **The addition formula *for* Mulatu numbers**.

Proof: By Corollary 4 above, we have,

$$M_n = F_{n-3} + F_{n-1} + F_{n+2}.$$

Hence it follows that

$$M_{n+k} = F_{n+k-3} + F_{n+k-1} + M_{n+k+2}.$$

Now using the addition formula for **Fibonacci** numbers given above, it follows that

$$M_{n+k} = (F_{n-1}F_{k-3} + F_n F_{k-2}) + (F_{n-1}F_{k-1} + F_n F_k) + (F_{n-1}F_{k+2} + F_n F_{k+3})$$

$$= (F_{n-1}F_{k-3} + F_{n-1} + F_{k-1} + F_{n-1}F_{k+2}) + (F_n F_{k-2} + F_n F_k + F_n F_{k+3})$$

$$= F_{n-1}(F_{k-3} + F_{k-1} + F_{k+2}) + F_n(F_{k-2} + F_k + {}_n F_{k+3})$$

$$= F_{n-1} M_k + F_n M_{k+1}.$$

Hence the theorem is proved.

References:

1. Mulatu Lemma, The Mulatu Numbers, *Advances and Applications in Mathematical Sciences,* Volume 10, issue 4,august 2011, page 431-440.

2. Burton, D. M., *Elementary number theory*. New York City, New York: McGraw-Hill. 1998.