# Modified Adomian Decomposition Method for Solving Fractional Singular Boundary Value Problems of Higher-Order Ordinary Differential Equations

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**Abstract**

This paper is an attempt to solve fractional singular boundary value problems of higher-order ordinary differential equations by formulating a new modification of the Adomian decomposition method (MADM). The proposed method can be applied to both to linear and nonlinear problems. Therefore, it is tested with some examples and the obtained results were noticed to approximate the exact solutions. It can be concluded that this formulated modification of ADM is efficient and reliable to solve such kinds of problems.

## 1 Introduction

Fractional differential equations are essential in numerous fields like fluid mechanics, biology, physics, engineering and electrical networks [1,2].

The decomposition method has been proven to be effective easy and accurate method to solve many types of equations; linear, nonlinear, ordinary, partial, deterministic or stochastic differential equations as in [3-5,9-15]. Many researchers found that this method arrives at approximate solutions which converge rapidly to accurate solutions. The aim of this work is to present a new reliable modification of Adomian decomposition method. For this reason, a new differential operator is proposed which can be used for singular boundary value problem. In this paper, we find solutions without massive computations and restrictive assumptions which change the physics problem into a mathematically tractable problem. This method is easier than the traditional methods. The theoretical treatment of the convergence of Adomian method has been investigated in[6–8,16].

## 2 Fractional Adomian decomposition method

Suppose the singular boundary value problem (SVP) of $(\alpha n+\alpha)$ order ordinary differential equation in the form

$$D_x^{(\alpha n+\alpha)}y+\frac{m\alpha}{x^\alpha}D_x^{(\alpha n)}y+Ny=f(x), \tag{1}$$

$$y(0)=b_0, D_x^\alpha y(0)=b_1, ..., D_x^{(\alpha n-\alpha)}y(0)=b_{n-1}, y(c)=b,$$

where $N$ is a nonlinear differential operator of order less than $n$, $f(x)$ is given function and $b_0; b_1; ...b_{n-1}, b, c$ are given constants.

We suggest the new differential operator, as below

$$L(.)=x^{-\alpha}\frac{d^{\alpha n}}{dx^{\alpha n}}x^{\alpha+\alpha n-\alpha m}\frac{d^\alpha}{dx^\alpha}x^{\alpha m-\alpha n}(.), \tag{2}$$

where $m\leq n, n\geq\alpha, 0<\alpha\leq 1$, the problem (1) can be written as

$$L_\alpha y=f(x)-Ny. \tag{3}$$

The $L^{-1}$ is therefor considered $n+1$ -fold integral operator, as below

$$L_\alpha^{-1}(.)=x^{\alpha n-\alpha m}\int_c^x x^{-\alpha-\alpha n+\alpha m}\int_0^x\int_0^x ... \int_0^x x^\alpha(.)dx^\alpha dx^\alpha ... dx^\alpha, \tag{4}$$

Applying $L_\alpha^{-1}$ on (3), we have

$$y(x)=\sigma(x)+L_\alpha^{-1}f(x)-L_\alpha^{-1}Ny, \tag{5}$$

where

$$L_\alpha(\sigma(x))=0.$$

The ADM represent the solution $y(x)$ and the non-linear function$Ny$ by infinite series

$$y(x)=\sum_{n=0}^{\infty}y_n(x), \tag{6}$$

and

$$Ny = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently by algorithm [9,13].

$A_n$ are the Adomain polynomials, which are obtain formula the following

$$A_0 = Z(y_0),$$
$$A_1 = y_1 Z'(y_0),$$
$$A_2 = y_2 Z'(y_0) + y_1^2 \frac{1}{2} Z''(y_0),$$

$$A_3 = y_3 Z'(y_0) + y_1 y_2 Z''(y_0) + y_1^3 \frac{1}{3!} Z'''(y_0), \tag{8}$$
$$\dots$$

Substituting eq.(6) and eq.(7) into eq.(5), we have

$$\sum_{n=0}^{\infty} y_n(x) = \sigma(x) + L_\alpha^{-1} f(x) - L_\alpha^{-1} \sum_{n=0}^{\infty} A_n, \tag{9}$$

we get the components $y_n$ can be specified as

$$y_0 = \sigma(x) + L_\alpha^{-1} f(x),$$
$$y_{n+1} = L_\alpha^{-1} A_n, \quad n \geq 0,$$

which gives

$$\begin{aligned} y_0 &= \sigma(x) + L_\alpha^{-1} f(x),\\ y_1 &= L_\alpha^{-1} A_0,\\ y_2 &= L_\alpha^{-1} A_1,\\ y_3 &= L_\alpha^{-1} A_2, \qquad (10)\\ &\dots \end{aligned}$$

From (8) and (10), we find the components $y_n(x)$, and hence the series solution of $y(x)$ in (6) can be directly obtained. For numerical aim, the $n-$ term approximate

$$\Psi(x) = \sum_{k=0}^{n-1} y_k,$$

can be used to approximate the exact solution.

# 3 Numerical Results

## 3.1 Example

Suppose the linear fractional (SVP)

$$D_x^{2\alpha}y + \frac{\rho}{x^\alpha}D_x^\alpha y = -x^{\alpha-\rho}cosx - (2\alpha-\rho)x^{-\rho}sinx, \tag{11}$$

$$y(0) = 0, y(1) = cos1,$$

where

$$L_\alpha(.) = x^{-\alpha}\frac{d^\alpha}{dx^\alpha}x^{2\alpha-\rho}\frac{d^\alpha}{dx^\alpha}x^{-\alpha+\rho}(.),$$

so

$$L_\alpha^{-1}(.) = x^{\alpha-\rho}\int_1^x x^{-2\alpha+\rho}\int_0^x x^\alpha(.)dx^\alpha dx^\alpha,$$

rewrite Eq.(11) as

$$L_\alpha y = -x^{\alpha-\rho}cosx - (2\alpha-\rho)x^{-\rho}sinx,$$

Take $L_\alpha^{-1}$ to both ides above Eq., we have

$$y = x^{\alpha-\rho}cos1 + L_\alpha^{-1}(-x^{\alpha-\rho}cosx - (2\alpha-\rho)x^{-\rho}sinx),$$

$$= x^{\alpha-\rho}cos1 + x^{\alpha-\rho}\int_1^x x^{-2\alpha+\rho}\int_0^x x^\alpha(-x^{\alpha-\rho}cosx - (2\alpha-\rho)x^{-\rho}sinx)dx^\alpha dx^\alpha,$$

implies

$$y(x) = x^{\alpha-\rho}cos1 - x^{\alpha-\rho}cos1 + x^{\alpha-\rho}cosx = x^{\alpha-\rho}cosx,$$

when put $\alpha = 1$, [17] and so, the right solution is too easily get by this manner.

## 3.2 Example

Consider the non-linear boundary value problem:

$$D_x^{2\alpha}y - \frac{\alpha}{x^\alpha}D_x^\alpha y = \frac{4\alpha x^{2\alpha}}{4+x^{2\alpha}}e^y, \tag{12}$$

$$y(0) = ln(\frac{\alpha}{4}), y(1) = ln(\frac{4\alpha}{5}),$$

applying

$$L_\alpha(.) = x^{-\alpha}\frac{d}{dx}x^{3\alpha}\frac{d}{dx}x^{-2\alpha}(.),$$

and

$$L_\alpha^{-1}(.) = x^{2\alpha} \int_1^x x^{-3\alpha} \int_0^x x^\alpha (.) dx^\alpha dx^\alpha,$$

in an operator form Eq.(12), we have

$$L_\alpha y = \frac{4\alpha x^{2\alpha}}{4 + x^{2\alpha}} e^y, \tag{13}$$

putting $L_\alpha^{-1}$ on both sides of Eq.(13), we get

$$y = y(0) + (y(1) + y(0))x^{2\alpha} + 4L_\alpha^{-1} \frac{\alpha x^{2\alpha}}{4 + x^{2\alpha}} e^y,$$

By MADM [9], we have

$$y_0 = Ln(\frac{\alpha}{4}),$$

$$y_1 = ln(\frac{4\alpha}{5})x^{2\alpha} + 4L_\alpha^{-1} \frac{\alpha x^{2\alpha}}{4 + x^{2\alpha}} A_0,$$

$$y_{n+1} = 4L_\alpha^{-1} \frac{\alpha x^{2\alpha}}{4 + x^{2\alpha}} A_n, n \geq 1.$$

Then

$y_1 = -0.0289294\alpha x^2 + 0.03125\alpha x^4 - 0.00260417\alpha x^6 + 0.000325521\alpha x^8 - 0.0000488281\alpha x^{10}$,

$y_2 = 0.0029882\alpha x^2 - 0.0105030\alpha x^6 + 0.0006510\alpha x^8 + 0.0000326\alpha x^{10} + 2.712673610^{-6}\alpha x^{12}$,

this means that the solution in a series form is given by

$$y = y_0 + y_1 + y_2$$

**Table 1. Compare between the exact solution with the approximate MADM in [0,1].**

| x | Approximate solution MADM $\alpha = 1$, | $\alpha = 0.99$ | Exact $\alpha = 1$ | Error $y_{Ex} - y_{MADM}$ |
|---|---|---|---|---|
| 0.0 | -1.38629, | -1.39634 | -1.38629 | 0.00000 |
| 0.1 | -1.38881, | -1.39896 | -1.38879 | 0.00002 |
| 0.2 | -1.39633, | -1.406776 | -1.39624 | 0.00009 |
| 0.3 | -1.40873, | -1.419660 | -1.40854 | 0.00016 |
| 0.4 | -1.42584, | -1.437460 | -1.42552 | 0.00032 |
| 0.5 | -1.44740, | -1.459910 | -1.44692 | 0.00018 |
| 0.6 | -1.47311, | -1.486710 | -1.47247 | 0.00062 |
| 0.7 | -1.50260, | -1.517500 | -1.50185 | 0.00059 |
| 0.8 | -1.53545, | -1.55187 | -1.53471 | 0.00026 |
| 0.9 | -1.57123, | -1.58938 | -1.57070 | 0.00116 |
| 1.0 | -1.60944, | -1.62955 | -1.60944 | 0.00000 |

# 4 Conclusion

In this work, we have presented MADM for solving fractional singular boundary value problems. The given examples and the derived results illustrate the advantages of using the method proposed in this work for these kinds of equations. Finally, the MADM is effective and efficient in finding the analytical solutions for a wide class of initial value problems.

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