GENERALIZED HYERS-ULAM -RASSISA STABILITY OF AN ADDITIVE ($\beta 1;\beta 2$)-FUNCTIONAL INEQUALITIES WITH THREE VARIABLES IN COMPLEX BANACH SPACE

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Abstract. In this paper we study to solve the of additive **($\beta 1;\ \beta 2$)**-functional inequality with three-variables and their Hyers-Ulam stability. First are investigated complex Banach spaces with a fixed point method and last are investigated in complex Banach spaces with a direct method. Then Hyers-Ulam stability of these equation are given and proven. These are the main results of this paper.

Keywords: Additive **($\beta 1;\ \beta 2$)**-Functional inequality; Fixed Point Method; Direct Method; Banach Space; Hyers-Ulam Stability.

Mathematics Subject Classification : $46S10;\ 39B62;\ 39B52;\ 47H10$.

I. INTRODUCTION

Let **X** and **Y** be a normed spaces on the same field K, and $f:\mathbf{X}\to\mathbf{Y}$. We use the notation $\cdot$ for all the norm on both **X** and **Y**. In this paper, we investisgate additive ($\beta 1;\ \beta 2$)- functional inequality when **X** be a real or complex normed space and **Y** a complex Banach spaces. We solve and prove the Hyers-Ulam stability of forllowing additive ($\beta 1;\ \beta 2$)-functional inequality

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}+\frac{z}{4}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right\|_{\mathbb{Y}} &\leq \left\|\beta_1\left(f\left(x+y+\frac{z}{2}\right)+f\left(x-y-\frac{z}{2}\right)-2f(x)\right)\right\|_{\mathbb{Y}}\\ &+\left\|\beta_2\left(f\left(x+y+\frac{z}{2}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right)\right\|_{\mathbb{Y}} \qquad (1.1)\end{aligned}$$

We solve and prove the Hyers-Ulam stability of forllowing additive $|\beta 1|+|\beta 1|<1$.
Note that in the preliminaries we just recap some of the most essential properties for the above problem and for the specific problem, please see the document. The HyersUlam stability was first investigated for functional equation of Ulam in [28] concerning the stability of group omomorphisms.
The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The Hyers [13] gave firts affirmative partial answer to the equation of Ulam in Banachspaces. After that, Hyers'Theorem was generalized by Aoki[1] additive mappings and by Rassias [26] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [10] by replacing the unboundedCauchy difference by ageneral control function in the spirit of Rassias' approach.The stability of quadratic functional equation was proved by Skof [27] for mappings$f : X \to Y$, where X is a normed space and Y is a Banach space. Park [24],[25] definedadditive γ -functional inequalities and proved the HyersUlam stability of the additiveγ -functional inequalities in Banach spaces and nonArchimedean Banach spaces. Thestability problems of various functional equations have been extensively investigated by a number of authors on the world. We recall a fundamental result in fixed point theory.Recently, in [3],[4],[21],[22],[24],[25] the authors studied the Hyers-Ulam stability for the following functional inequalities:

$$\left\|f(\frac{x+y}{2}+z)-f(\frac{x+y}{2})-f(z)\right\| \leq \left\|f(\frac{x+y}{2^2}+\frac{z}{2})-\frac{1}{2}f(\frac{x+y}{2})-\frac{1}{2}f(z)\right\| \tag{1.2}$$

$$\left\|f(\frac{x+y}{2^2}+\frac{z}{2})-\frac{1}{2}f(\frac{x+y}{2})-\frac{1}{2}f(z)\right\| \leq \left\|f(\frac{x+y}{2}+z)-f(\frac{x+y}{2})-f(z)\right\| \tag{1.3}$$

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \left\|\rho\Big(2f(\frac{x+y}{2})-f(x)-f(y)\Big)\right\| \tag{1.4}$$

$$\left\|2f\Big(\frac{x+y}{2}\Big)-f(x)-f(y)\right\| \leq \left\|\rho\big(f(x+y)-f(x)-f(y)\big)\right\| \tag{1.5}$$

and

$$\begin{aligned} &\left\|f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+y}{2}-z\Big)-2f\Big(\frac{x+y}{2}\Big)-2f(z)\right\| \\ \leq\ &\left\|\beta\Big(2f\Big(\frac{x+y}{2^2}+\frac{z}{2}\Big)+2f\Big(\frac{x+y}{2^2}-\frac{z}{2}\Big)-f\Big(\frac{x+y}{2}\Big)-f(z)\Big)\right\| \end{aligned} \tag{1.6}$$

$$\begin{aligned} &\left\|2f\Big(\frac{x+y}{2^2}+\frac{z}{2}\Big)+2f\Big(\frac{x+y}{2^2}-\frac{z}{2}\Big)-f\Big(\frac{x+y}{2}\Big)-f(z)\right\| \\ \leq\ &\left\|\beta\Big(f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+y}{2}-z\Big)-2f\Big(\frac{x+y}{2}\Big)-2f(z)\Big)\right\| \end{aligned} \tag{1.7}$$

finaly

$$\begin{aligned} \left\|f(x+y)-f(x)-f(y)\right\| \leq &\left|\beta_1\Big(f(x+y)+f(x-y)-2f(x)\Big)\right\| \\ &+\left\|\beta_2\Big(2f\Big(\frac{x+y}{2}\Big)-f(x)-f(y)\Big)\right\| \end{aligned} \tag{1.8}$$

in complex Banach spaces

In this paper, we solve and proved the Hyers-Ulam stability for $(\beta 1;\beta 2)$-functional inequalities (1.1), ie the $(\beta 1;\beta 2)$-functional inequalities with three variables. Under suitable assumptions on spaces X and Y, we will prove that the mappings satisfying the $(\beta 1;\beta 2)$-functional inequatilies (1.1). Thus, the results in this paper are generalization of those in [3],[4],[14],[21] for $(\beta 1;\beta 2)$-functional inequatilies with three variables.The paper is organized as followns:In section preliminarier we remind some basic notations in [3,7] such as complete generalized metric space and Solutions of the inequalities.

Section 3: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the addive $(\beta 1;\beta 2)$- functional inequalities (1.1) when X be a real or complete normed space and Y complex Banach space.

Section 4: In this section, I use the method of directly determining the solution for (1.1) when X be a real or complete normed space and Y complex Banach space.

2. Preliminaries

2.1. Complete Generalized Metric Space And Solutions of The Equalities.

Theorem 2.1. *Let $X; d$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(Jn; Jn+1)= 1$$

for all nonegative integers n or there exists a positive integer n0 such that

(1) $d\big(J^n, J^{n+1}\big) < \infty$, $\forall n \geq n_0$;
(2) The sequence $\big\{J^n x\big\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \big\{y \in X | d\big(J^n, J^{n+1}\big) < \infty\big\}$;
(4) $d\big(y, y^*\big) \leq \frac{1}{1-l} d\big(y, Jy\big)$ $\forall y \in Y$

2.2. Solutions of the inequalities.

The functional equation

$$f(x + y) = f(x) + f(y)$$

is called the Cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

3. Establish The Solution of The Additive $(\beta 1;\beta 2)$-Function Inequalities Using a Fixed Point Method

Now, we first study the solutions of (1.1). Note that for these inequalities, when X be a real or complete normed space and Y complex Banach space

Lemma 3.1. *A mapping* $f:\mathbb{X}\to\mathbb{Y}$ *satisfies* $f\big(0\big)=0$ *and*

$$\begin{aligned}\left\|2f\Big(\frac{x+y}{2}+\frac{z}{4}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\right\|_{\mathbb{Y}}&\\ \le\left\|\beta_1\Big(f\Big(x+y+\frac{z}{2}\Big)+f\Big(x-y-\frac{z}{2}\Big)-2f(x)\Big)\right\|_{\mathbb{Y}}&\\ +\left\|\beta_2\Big(f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\Big)\right\|_{\mathbb{Y}}&\qquad(3.1)\end{aligned}$$

for all x,y,z$\in X$ *if and only if* $f:\mathbb{X}\to\mathbb{Y}$ *is additive*

Proof. Assume that $f:\mathbb{X}\to\mathbb{Y}$ satisfies (3.1)
Letting $y=z=0$ in (3.1), we get

$$\left\|2f\Big(\frac{x}{2}\Big)-f\Big(x\Big)\right\|_{\mathbb{Y}}\le 0$$

Thus

$$f\Big(\frac{x}{2}\Big)=\frac{1}{2}f(x)\qquad(3.2)$$

for all $x\in\mathbb{X}$ It follows from (3.1) and (3.2) that

$$\begin{aligned}&\left\|f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\right\|_{\mathbb{Y}}\\ &=\left\|2f\Big(\frac{x+y}{2}+\frac{z}{4}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\right\|_{\mathbb{Y}}\\ &\le\left\|\beta_1\Big(f\Big(x+y+\frac{z}{2}\Big)+f\Big(x-y-\frac{z}{2}\Big)-2f(x)\Big)\right\|_{\mathbb{Y}}\\ &+\left\|\beta_2\Big(f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\Big)\right\|_{\mathbb{Y}}\qquad(3.3)\end{aligned}$$

and so

$$\begin{aligned}&\Big(1-\Big|\beta_2\Big|\Big)\left\|f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\right\|_{\mathbb{Y}}\\ &\le\Big|\beta_1\Big|\left\|f\Big(x+y+\frac{z}{2}\Big)+f\Big(x-y-\frac{z}{2}\Big)-2f(x)\right\|_{\mathbb{Y}}\qquad(3.4)\end{aligned}$$

Letting $u=x+y+\frac{z}{2},v=x-y-\frac{z}{2}$ in (3.4), we get

$$\Big(1-\big|\beta_2\big|\Big)\bigg\|f\Big(u\Big)-f\Big(\frac{u+v}{2}\Big)-f\Big(\frac{u-v}{2}\Big)\bigg\|_{\mathbb{Y}} \leq \big|\beta_1\big|\bigg\|f\Big(u\Big)+f\Big(v\Big)-2f\Big(\frac{u+v}{2}\Big)\bigg\|_{\mathbb{Y}} \tag{3.5}$$

for all $u, v \in \mathbb{X}$

and so

$$\frac{1}{2}\Big(1-\big|\beta_2\big|\Big)\bigg\|f\Big(u+v\Big)+f\Big(u-v\Big)-2f\Big(u\Big)\bigg\|_{\mathbb{Y}} \leq \big|\beta_1\big|\bigg\|f\Big(u+v\Big)-f\Big(u\Big)-f\Big(v\Big)\bigg\|_{\mathbb{Y}} \tag{3.6}$$

for all $u, v \in X$ It follows from (3.4) and (3.5) that

$$\frac{1}{2}\Big(1-\big|\beta_2\big|\Big)^2\bigg\|f\Big(x+y+\frac{z}{2}\Big)-f(x)+f\Big(y+\frac{z}{2}\Big)\bigg\|_{\mathbb{Y}} \leq \big|\beta_1\big|^2\bigg\|f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\bigg\|_{\mathbb{Y}} \tag{3.7}$$

Since $\sqrt{2}\big|\beta_1\big|+\big|\beta_2\big|<1$

and so

$$f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)=0$$

. for all $x, y, z \in X$. Thus f is additive. □

Theorem 3.2. *suppose $\varphi : \mathbb{X}^3 \to \big[0,\infty\big)$ be a function such that there exists an $L<1$ with*

$$\varphi\Big(\frac{x}{2},\frac{y}{2},\frac{z}{2}\Big) \leq \frac{L}{2}\varphi(x,y,z) \tag{3.8}$$

for all $x, y, z \in \mathbb{X}$. Give $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f\big(0\big)=0$ and

$$\begin{aligned}\bigg\|2f\Big(\frac{x+y}{2}+\frac{z}{4}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\bigg\|_{\mathbb{Y}} &\leq \bigg\|\beta_1\Big(f\Big(x+y+\frac{z}{2}\Big)+f\Big(x-y-\frac{z}{2}\Big)-2f(x)\Big)\bigg\|_{\mathbb{Y}} \\ &+\bigg\|\beta_2\Big(f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\Big)\bigg\|_{\mathbb{Y}}+\varphi\big(x,y,z\big)\end{aligned} \tag{3.9}$$

for all $x, y, z \in \mathbb{X}$.
Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\Big\|f\big(x\big)-H\big(x\big)\Big\|_{\mathbb{Y}} \leq \frac{1}{1-L}\varphi\big(x,0,0\big) \tag{3.10}$$

for all $x \in \mathbb{X}$

Proof. Letting $y = z = 0$ in (3.9), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_{\mathbb{Y}} \leq \varphi(x, 0, 0) \tag{3.11}$$

for all $x \in \mathbb{X}$.
Consider the set

$$\mathbb{S} := \Big\{ h : \mathbb{X} \to \mathbb{Y}, h(0) = 0 \Big\}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d\Big(g, h\Big) := inf \Big\{ \lambda \in \mathbb{R}_+ : \Big\| g(x) - h(x) \Big\| \leq \lambda\varphi(x, 0, 0), \forall x \in X \Big\},$$

where, as usual, $inf \phi = +\infty$. It easy to show that $\Big(\mathbb{S}, d\Big)$ is complete $\Big(see[16]\Big)$ Now we cosider the linear mapping $J : \mathbb{S} \to \mathbb{S}$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in \mathbb{X}$. Let $g, h \in \mathbb{S}$ be given such that $d(g, h) = \epsilon$ then

$$\Big\| g(x) - h(x) \Big\| \leq \epsilon\varphi(x, 0, 0,)$$

for all $x \in \mathbb{X}$.
Hence

$$\begin{aligned} \Big\| Jg(x) - Jhf(x) \Big\| &= \left\| 2g\left(\frac{x}{2}\right) - 2hf\left(\frac{x}{2}\right) \right\| \leq 2\epsilon\varphi\left(\frac{x}{2}, 0, 0\right) \\ &\leq 2\epsilon\frac{L}{2}\varphi(x, 0, 0) \leq L\epsilon\varphi(x, 0, 0) \end{aligned}$$

for all $x \in \mathbb{X}$. So $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L \cdot \epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in X$ .It follows from (3) that $d(f.Jf) \leq 1$. Follow Theorem 1.1, there exists a mapping $H : \mathbb{X} \to \mathbb{Y}$ satifying the following:

(1) H is a fixed point of J, ie.,

$$H(x) = 2H\left(\frac{x}{2}\right) \tag{3.12}$$

for all $x \in X$. The mapping H is a unique fixed point J in the set

$$\mathbb{M} = \Big\{ g \in \mathbb{S} : d(f, g) < \infty \Big\}$$

This implies that H is a unique mapping satisfying (3.12) such that there exists a $\lambda \in \Big(0, \infty\Big)$ satisfying

$$\Big\| f(x) - H(x) \Big\| \leq \lambda\varphi(x, 0, 0)$$

for all $x \in \mathbb{X}$

(2) $d\Big(J^l f, H\Big) \to 0$ as $l \to \infty$. This implies equality

$$\lim_{l\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) = H(x)$$

for all $x \in \mathbb{X}$

(3) $d\Big(f, H\Big) \leq \frac{1}{1-L} d\Big(f, Jf\Big)$. which implies

$$\Big\|f(x) - H(x)\Big\| \leq \frac{L}{1-L}\varphi(x,0,0)$$

for all $x \in X$. It follows (3.8) and (3.9) that

$$\begin{aligned}
&\Bigg\|2H\Big(\frac{x+y}{2} + \frac{z}{4}\Big) - H(x) - H\Big(y + \frac{z}{2}\Big)\Bigg\|_{\mathbb{Y}} \\
&= \lim_{n\to\infty} 2^n \Bigg\|2f\Big(\frac{x+y}{2^{n+1}} + \frac{z}{2^{n+2}}\Big) - f\Big(\frac{x}{2^n}\Big) - f\Big(\frac{y}{2^n} + \frac{z}{2^{n+1}}\Big)\Bigg\|_{\mathbb{Y}} \\
&\leq \lim_{n\to\infty} 2^n \Big|\beta_1\Big| \Bigg\|f\Big(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\Big) + f\Big(\frac{x-y}{2^n} - \frac{z}{2^{n+1}}\Big) - 2f\Big(\frac{x}{2^n}\Big)\Bigg\|_{\mathbb{Y}} \\
&+ \lim_{n\to\infty} 2^n \Big|\beta_2\Big| \Bigg\|f\Big(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\Big) + f\Big(\frac{x}{2^n}\Big) - f\Big(\frac{y}{2^n} + \frac{z}{2^{n+1}}\Big)\Bigg\|_{\mathbb{Y}} + \lim_{n\to\infty} 2^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) \\
&= \Bigg\|\beta_1\Big(H\Big(x+y+\frac{z}{2}\Big) + H\Big(x-y-\frac{z}{2}\Big) - 2H(x)\Big)\Bigg\|_{\mathbb{Y}} + \Bigg\|\beta_2\Big(H\Big(x+y+\frac{z}{2}\Big) \\
&- H(x) - H(y + \frac{z}{2})\Big)\Bigg\|_{\mathbb{Y}}
\end{aligned}$$

for all $x, y, z \in \mathbb{X}$. So

$$\begin{aligned}
&\Bigg\|2H\Big(\frac{x+y}{2} + \frac{z}{4}\Big) - H(x) - H\Big(y + \frac{z}{2}\Big)\Bigg\|_{\mathbb{Y}} \\
&\qquad \leq \Bigg\|\beta_1\Big(H\Big(x+y+\frac{z}{2}\Big) + H\Big(x-y-\frac{z}{2}\Big) - 2H(x)\Big)\Bigg\|_{\mathbb{Y}} \\
&\qquad + \Bigg\|\beta_2\Big(H\Big(x+y+\frac{z}{2}\Big) - H(x) - H\Big(y+\frac{z}{2}\Big)\Big)\Bigg\|_{\mathbb{Y}}
\end{aligned}$$

for all $x, y, z \in X$. By *Lemma* 3.1, the mapping $H : \mathbb{X} \to \mathbb{Y}$ is additive.

□

Theorem 3.3. *Let $\varphi : \mathbb{X}^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \leq 2L\varphi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big) \tag{3.13}$$

for all $x, y, z \in \mathbb{X}$*. Let* $f : \mathbb{X} \to \mathbb{Y}$ *be a mapping satisfy* $f(0) = 0$ *and*

$$\begin{aligned}\Big\|2f\Big(\frac{x+y}{2}+\frac{z}{4}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\Big\|_{\mathbb{Y}} &\\ \le \Big\|\beta_1\Big(f\Big(x+y+\frac{z}{2}\Big)+f\Big(x-y-\frac{z}{2}\Big)-2f(x)\Big)\Big\|_{\mathbb{Y}} &\\ + \Big\|\beta_2\Big(f\Big(x+y+\frac{z}{2}\Big)-f(x)-f\Big(y+\frac{z}{2}\Big)\Big)\Big\|_{\mathbb{Y}}+\varphi(x,y,z) & \qquad (3.14)\end{aligned}$$

for all $x, y, z \in \mathbb{X}$*.*
Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\Big\|f(x)-H(x)\Big\|_{\mathbb{Y}} \le \frac{L}{1-L}\varphi(x,0,0) \qquad (3.15)$$

for all $x \in \mathbb{X}$

Proof. Letting $y = z = 0$ in (3.14), we get

$$\Big\|2f\big(\frac{x}{2}\big)-f(x)\Big\|_{\mathbb{Y}} \le \varphi(x,0,0) \qquad (3.16)$$

and so

$$\Big\|f(x)-\frac{1}{2}f\Big(2x\Big)\Big\|_{\mathbb{Y}} \le \frac{1}{2}\varphi\Big(2x,0,0\Big) \le L\varphi(x,0,0) \qquad (3.17)$$

for all $x \in \mathbb{X}$.

Let $\Big(S, d\Big)$ be the generalized metric space defined in the proof of Theorem 3.2
Now we consider the mapping $J : \mathbb{S} \to \mathbb{S}$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.
It follows (3.17) that $d\Big(f, Jf\Big) \le \frac{1}{2}\varphi(2x, 0, 0) \le L\varphi(x, 0, 0)$. So

for all $x \in X$.
The rest of the proof is similar to proof of Theorem 3.2. □

From proving the theorems we have consequences:

Corollary 3.4. *Let* $r > 1$ *and* θ *be nonnegative real numbers and let* $f : \mathbb{X} \to \mathbb{Y}$ *be a mapping satisfy* $f(0) = 0$ *and*

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}+\frac{z}{4}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right\|_{\mathbb{Y}}&\\ \leq \left\|\beta_1\left(f\left(x+y+\frac{z}{2}\right)+f\left(x-y-\frac{z}{2}\right)-2f(x)\right)\right\|_{\mathbb{Y}}&\\ +\left\|\beta_2\left(f\left(x+y+\frac{z}{2}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right)\right\|_{\mathbb{Y}}+\theta(\|x\|^r+\|y\|^r+\|z\|^r)&\end{aligned} \tag{3.18}$$

for all $x, y, z \in \mathbb{X}$.
Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\|f(x)-H(x)\right\|_{\mathbb{Y}} \leq \frac{2^r\theta}{2^r-2}\varphi\|x\|^r \tag{3.19}$$

for all $x \in \mathbb{X}$

Corollary 3.5. *Let* $r < 1$ *and* θ *be nonnegative real numbers and let* $f : \mathbb{X} \to \mathbb{Y}$ *be a mapping satisfy* $f(0) = 0$ *and*

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}+\frac{z}{4}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right\|_{\mathbb{Y}}&\\ \leq \left\|\beta_1\left(f\left(x+y+\frac{z}{2}\right)+f\left(x-y-\frac{z}{2}\right)-2f(x)\right)\right\|_{\mathbb{Y}}&\\ +\left\|\beta_2\left(f\left(x+y+\frac{z}{2}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right)\right\|_{\mathbb{Y}}+\theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)&\end{aligned} \tag{3.20}$$

for all $x, y, z \in \mathbb{X}$.
Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\|f(x)-H(x)\right\|_{\mathbb{Y}} \leq \frac{2^r\theta}{2-2^r}\|x\|^r \tag{3.21}$$

for all $x \in \mathbb{X}$

4. Establish The Solution of The Additive ($\beta 1; \beta 2$)-Function Inequalities Using A Aireect Method

Next, we study the solutions of (1.1) . Note that for these inequalities, when X be a real or complete normed space and Y complex Banach space.

Theorem 4.1. *Let* $\varphi : \mathbb{X}^3 \to [0, \infty)$ *be a function such that*

$$\psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{4.1}$$

and let $f : \mathbb{X} \to \mathbb{Y}$ *be a mapping* $f(0) = 0$ *and*

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}+\frac{z}{4}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right\|_{\mathbb{Y}} \\
&\qquad \le \left\|\beta_1\left(f\left(x+y+\frac{z}{2}\right)+f\left(x-y-\frac{z}{2}\right)-2f(x)\right)\right\|_{\mathbb{Y}} \\
&\qquad + \left\|\beta_2\left(f\left(x+y+\frac{z}{2}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right)\right\|_{\mathbb{Y}}+\varphi(x,y,z) \qquad (4.2)
\end{aligned}$$

for all $x,y,z\in\mathbb{X}$.
Then there exists a unique additive mapping $H:\mathbb{X}\to\mathbb{Y}$ *such that*

$$\left\|f(x)-H(x)\right\|_{\mathbb{Y}}\le\phi\left(x,0,0\right) \qquad (4.3)$$

for all $x\in\mathbb{X}$

Proof. Let $y=z=0$ in (4.9), we get

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|_{\mathbb{Y}}\le\varphi\left(x,0,0\right) \qquad (4.4)$$

for all $x\in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_{\mathbb{Y}}\le\varphi(x,0,0) \qquad (4.5)$$

for all $x\in\mathbb{X}$. Hence

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|_{\mathbb{Y}} \\
&\qquad \le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbb{Y}} \\
&\qquad \le \sum_{j=l}^{m-1}2^j\varphi\left(\frac{x}{2^j},0,0\right) \qquad (4.6)
\end{aligned}$$

for all nonnegative integers m and l with $m>l$ and all $x\in\mathbb{X}$. It follows from (4.6)that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ coverges. So one can define the mapping $H:\mathbb{X}\to\mathbb{Y}$ by

$$H(x):=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right) \qquad (4.7)$$

It follows from (4.1) and (4.9) that

Theorem 4.2. *Let $\varphi : \mathbb{X}^3 \to [0, \infty)$ be a function such that*

$$\psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{4.10}$$

and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping $f(0) = 0$ and

$$\begin{aligned} &\left\| 2f\left(\frac{x+y}{2} + \frac{z}{4}\right) - f(x) - f\left(y + \frac{z}{2}\right) \right\|_{\mathbb{Y}} \\ &\qquad \leq \left\| \beta_1 \left(f\left(x + y + \frac{z}{2}\right) + f\left(x - y - \frac{z}{2}\right) - 2f(x) \right) \right\|_{\mathbb{Y}} \\ &\qquad + \left\| \beta_2 \left(f\left(x + y + \frac{z}{2}\right) - f(x) - f\left(y + \frac{z}{2}\right) \right) \right\|_{\mathbb{Y}} + \varphi(x, y, z) \end{aligned} \tag{4.11}$$

for all $x, y, z \in \mathbb{X}$.
Then there exists a unique additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\| f(x) - H(x) \right\|_{\mathbb{Y}} \leq \psi\left(x, 0, 0\right) \tag{4.12}$$

for all $x \in \mathbb{X}$

Proof. Let $y = z = 0$ in (4.11), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_{\mathbb{Y}} \leq \varphi\left(x, 0, 0\right) \tag{4.13}$$

for all $x \in X$. So

$$\left\| f(x) - \frac{1}{2} f\left(2x\right) \right\|_{\mathbb{Y}} \leq \frac{1}{2} \varphi(2x, 0, 0) \tag{4.14}$$

for all $x \in \mathbb{X}$. Hence

$$\begin{aligned} &\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\|_{\mathbb{Y}} \\ &\qquad \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\|_{\mathbb{Y}} \\ &\qquad \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi\left(2^j x, 0, 0\right) \end{aligned} \tag{4.15}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (4.15)that the sequence $\left\{ \frac{1}{2^n} f\left(2^n x\right) \right\}$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{ \frac{1}{2^n} f\left(2^n x\right) \right\}$ coverges. So one can define the mapping $H : \mathbb{X} \to \mathbb{Y}$ by

$$H(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right) \tag{4.16}$$

$x \in \mathbb{X}$. The rest of the proof is similar to the proof of Theorem 4.1. □

$$\begin{aligned}
&\left\|2H\Big(\frac{x+y}{2}+\frac{z}{4}\Big)-H(x)-H\Big(y+\frac{z}{2}\Big)\right\|_{\mathbb{Y}} \\
&= \lim_{n\to\infty} 2^n\left\|2f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^{n+2}}\right)-f\Big(\frac{x}{2^n}\Big)-f\Big(\frac{y}{2^n}+\frac{z}{2^{n+1}}\Big)\right\|_{\mathbb{Y}} \\
&\leq \lim_{n\to\infty} 2^n\Big|\beta_1\Big|\left\|f\Big(\frac{x+y}{2^n}+\frac{z}{2^{n+1}}\Big)+f\Big(\frac{x-y}{2^n}-\frac{z}{2^{n+1}}\Big)-2f\Big(\frac{x}{2^n}\Big)\right\|_{\mathbb{Y}} \\
&+ \lim_{n\to\infty} 2^n\Big|\beta_2\Big|\left\|f\Big(\frac{x+y}{2^n}+\frac{z}{2^{n+1}}\Big)+f\Big(\frac{x}{2^n}\Big)-f\Big(\frac{y}{2^n}+\frac{z}{2^{n+1}}\Big)\right\|_{\mathbb{Y}} + \lim_{n\to\infty} 2^n\varphi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) \\
&= \left\|\beta_1\Big(H\Big(x+y+\frac{z}{2}\Big)+H\Big(x-y-\frac{z}{2}\Big)-2H(x)\Big)\right\|_{\mathbb{Y}} \\
&+ \left\|\beta_2\Big(H\Big(x+y+\frac{z}{2}\Big)-H(x)-H\Big(y+\frac{z}{2}\Big)\Big)\right\|_{\mathbb{Y}} \qquad (4.8)
\end{aligned}$$

for all $x, y, z \in \mathbb{X}$. So

$$\begin{aligned}
\left\|2H\Big(\frac{x+y}{2}+\frac{z}{4}\Big)-H(x)-H\Big(y+\frac{z}{2}\Big)\right\|_{\mathbb{Y}}
&\leq \left\|\beta_1\Big(H\Big(x+y+\frac{z}{2}\Big)+H\Big(x-y-\frac{z}{2}\Big)-2H(x)\Big)\right\|_{\mathbb{Y}} \\
&+ \left\|\beta_2\Big(H\Big(x+y+\frac{z}{2}\Big)-H(x)-H\Big(y+\frac{z}{2}\Big)\Big)\right\|_{\mathbb{Y}} \qquad (4.9)
\end{aligned}$$

$$H\Big(x+y+\frac{z}{2}\Big)-H(x)-H\Big(y+\frac{z}{2}\Big)=0$$

for all $x, y, z \in \mathbb{X}$. By *Lemma* 3.1, the mapping $H : \mathbb{X} \to \mathbb{Y}$ is additive.

Now, let $T : \mathbb{X} \to \mathbb{Y}$ be another additive mapping satisfying (4.3). Then we have

$$\begin{aligned}
\left\|H(x)-T(x)\right\|_{\mathbb{Y}} &= \left\|2^qH\Big(\frac{x}{2^q}\Big)-2^qT\Big(\frac{x}{2^q}\Big)\right\|_{\mathbb{Y}} \\
&\leq \left\|2^qH\Big(\frac{x}{2^q}\Big)-2^qf\Big(\frac{x}{2^q}\Big)\right\|_{\mathbb{Y}} + \left\|2^qT\Big(\frac{x}{2^q}\Big)-2^qf\Big(\frac{x}{2^q}\Big)\right\|_{\mathbb{Y}} \\
&\leq 2^{q+1}\psi\Big(\frac{x}{2^q},0,0\Big)
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in \mathbb{X}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbb{X}$. This proves the uniqueness of H. $\square$

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f(0) = 0$ and*

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}+\frac{z}{4}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right\|_{\mathbb{Y}} \\ \leq \left\|\beta_1\left(f\left(x+y+\frac{z}{2}\right)+f\left(x-y-\frac{z}{2}\right)-2f(x)\right)\right\|_{\mathbb{Y}} \\ + \left\|\beta_2\left(f\left(x+y+\frac{z}{2}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right)\right\|_{\mathbb{Y}} + \theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)\end{aligned} \tag{4.17}$$

for all $x, y, z \in \mathbb{X}$.
Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{2^r\theta}{2^r-2}\|x\|^r \tag{4.18}$$

for all $x \in \mathbb{X}$

Corollary 4.4. *Let $r < 1$ and θ be nonnegative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f(0) = 0$ and*

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}+\frac{z}{4}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right\|_{\mathbb{Y}} \\ \leq \left\|\beta_1\left(f\left(x+y+\frac{z}{2}\right)+f\left(x-y-\frac{z}{2}\right)-2f(x)\right)\right\|_{\mathbb{Y}} \\ + \left\|\beta_2\left(f\left(x+y+\frac{z}{2}\right)-f(x)-f\left(y+\frac{z}{2}\right)\right)\right\|_{\mathbb{Y}} + \theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)\end{aligned} \tag{4.19}$$

for all $x, y, z \in \mathbb{X}$.
Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{2^r\theta}{2-2^r}\|x\|^r \tag{4.20}$$

for all $x \in \mathbb{X}$

5. Conclusion

In this paper, I have shown that the solutions of the $\beta 1; \beta 2$-functional inequalities are additive mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [3,4,14, 21] .

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