ESTABLISH AN ADDITIVE (s; t)-FUNCTION INEQUALITIES BY FIXED POINT METHOD AND DIRECT METHOD WITH *n*-VARIABLES IN BANACH SPACE

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ABSTRACT.

In this paper we study to solve the of additive ***(s,t)-****functional inequality with n-variables and their Hyers-Ulam stability. First are investigated in Banach spaces with a fixed point method and last are investigated in Banachspaces with a direct method .These are the main results of this paper.*

KEYWORDS*:* *Additive (s; t)-functional inequality; fixed point method; direct method; Banach space; Hyers - Ulam stability.*

1. INTRODUCTION

Let X and **Y** be a normed spaces on the same field **K**, and $f: \mathbf{X} \to \mathbf{Y}$. We use the notation $\cdot$ for all the norm on both **X** and **Y**. In this paper, we investisgate additive **(s,t)-**functional inequality when **X** be a normed space and **Y** a Banach spaces. We solve and prove the Hyers-Ulam stability of forllowing additive **(s; t)-**functional inequality.

$$\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\right\|_{\mathbf{Y}}$$
$$\leq \left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}$$
$$+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}} \quad (1.1)$$

In which **(s;t)** are fixed nonzero complex numbers with **G**(s;t)-functional inequality.Note that in the preliminaries we just replacing some of the most essential properties for the above problem and for the specific problem, please see the document. The HyersUlam stability was first investigated for functional equation of Ulam in [29] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The Hyers [14] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers'Theorem was generalized by Aoki [1] additive mappings and byRassias [27] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [28] for mappings $f: \mathbf{X} \to \mathbf{Y}$, where **X** is a normed space and **Y** is a Banach space. Park [25],[26] defined additive γ -functional inequalities and proved the HyersUlam stability of the additive γ -functional inequalities in Banach spaces and nonArchimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors on the world. We recall a fundamental result in fixed point theory. Recently, in [3],[4],[22],[23],[25],[26] the authors studied the Hyers-Ulam stability for the following functional nequalities

$$\|f(\frac{x+y}{2}+z)-f(\frac{x+y}{2})-f(z)\| \leq \|f(\frac{x+y}{2^2}+\frac{z}{2})-\frac{1}{2}f(\frac{x+y}{2})-\frac{1}{2}f(z)\| \quad (1.2)$$

$$\|f(\frac{x+y}{2^2}+\frac{z}{2})-\frac{1}{2}f(\frac{x+y}{2})-\frac{1}{2}f(z)\| \leq \|f(\frac{x+y}{2}+z)-f(\frac{x+y}{2})-f(z)\| \quad (1.3)$$

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \left\|\rho\Big(2f\Big(\frac{x+y}{2}\Big)-f(x)-f(y)\Big)\right\| \quad (1.4)$$

$$\left\|2f(\frac{x+y}{2})-f(x)-f(y)\right\| \leq \left\|\rho(f(x+y)-f(x)-f(y))\right\| \quad (1.5)$$

and

$$\left\|f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+y}{2}-z\Big)-2f\Big(\frac{x+y}{2}\Big)-2f(z)\right\|$$
$$\leq \left\|\beta\Big(2f\Big(\frac{x+y}{2^2}+\frac{z}{2}\Big)+2f\Big(\frac{x+y}{2^2}-\frac{z}{2}\Big)-f\Big(\frac{x+y}{2}\Big)-f(z)\Big)\right\| \quad (1.6)$$

$$\left\|2f\Big(\frac{x+y}{2^2}+\frac{z}{2}\Big)+2f\Big(\frac{x+y}{2^2}-\frac{z}{2}\Big)-f\Big(\frac{x+y}{2}\Big)-f(z)\right\|$$
$$\leq \left\|\beta\Big(f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+y}{2}-z\Big)-2f\Big(\frac{x+y}{2}\Big)-2f(z)\Big)\right\| \quad (1.7)$$

and

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \left\|\beta_1(f(x+y)+f(x-y)-2f(x))\right\|$$
$$+\left\|\beta_2\Big(2f\Big(\frac{x+y}{2}\Big)-f(x)-f(y)\Big)\right\| \quad (1.8)$$

finaly

$$\left\|f\left(x_1+x_2+...+x_n\right)-f\left(x_1\right)-f\left(x_2+...+x_n\right)\right\|_{\mathbb{Y}}$$
$$\leq\left\|\beta_1\left(f\left(x_1+x_2+...+x_n\right)-f\left(x_1-x_2-...-x_n\right)-2f\left(x_1\right)\right)\right\|_{\mathbb{Y}}$$
$$+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+...+x_n}{2}\right)-f\left(x_1\right)-f\left(x_2+...+x_n\right)\right)\right\|_{\mathbb{Y}} \quad (1.9)$$

in Banach spaces
In this paper, we solve and proved the Hyers-Ulam stability for **(s, t)-**functional inequalities (1.1), ie the **(s;t)-**functional inequalities with three variables. Under suitable assumptions on spaces **X** and **Y**, we will prove that the mappings satisfying the **(s; t)-**functional inequatilies (1.1). Thus, the results in this paper are generalization of those in [3],[4],[5],[15],[22] for **(s; t)-**functional inequatilies with three variables.
The paper is organized as followns: In section preliminarier we remind some basic notations in [3],[8] such as complete generalized metric space and Solutions of the inequalities.

Section 3: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the addive **(s,t)-** functional inequalities (1.1) when **X** be a normed space and **Y** Banach space.

Section 4: In this section, I use the method of directly determining the solution for (1.1) when **X** be a normed space and **Y** Banach space.

2. Preliminaries

2.1. Complete Generalized Metric Space And Solutions of The Inequalities.

Theorem 2.1. *Let $\left(X,d\right)$ be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d\left(J^n,J^{n+1}\right)=\infty$$

for all nonegative integers n or there exists a positive integer n_0 such that

(1) $d\left(J^n,J^{n+1}\right)<\infty,\ \forall n\geq n_0$;
(2) The sequence $\left\{J^nx\right\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y=\left\{y\in X|d\left(J^n,J^{n+1}\right)<\infty\right\}$;
(4) $d\left(y,y^*\right)\leq\frac{1}{1-l}d\left(y,Jy\right)\ \forall y\in Y$

2.2. Solutions of The Inequalities.

The functional equation f(x + y) = f(x) + f(y) is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an additive mapping. For convenience, I also require the following classes of mappings:

$$F_0\left(\mathbf{X},\mathbf{Y}\right)=\left\{f:\mathbf{X}\to\mathbf{Y}:f\left(0\right)=0\right\}$$

$$A\left(\mathbf{X},\mathbf{Y}\right)=\left\{f:\mathbf{X}\to\mathbf{Y}\right\}$$

$$A_0\left(\mathbf{X},\mathbf{Y}\right)=F_0\left(\mathbf{X},\mathbf{Y}\right)\cap A\left(\mathbf{X},\mathbf{Y}\right)$$

$$\left(\mathbb{C}\setminus\{0\},\mathbf{Y}\right)=\{G:\mathbb{C}\setminus\{0\}\to\mathbf{Y},G\left(s,t\right)=\sqrt{2}\left|s\right|+\left|t\right|<1\}$$

3. Establish The solution of The Additive (s,t)-Function Inequalities Using A Fixed Foint Method

Now, we first study the solutions of (1.1). Note that for these inequalities, when **X** be a normed space and **Y** is a Banach space.

Lemma 3.1. *If a mapping $f \in A\Big(\mathbf{X},\mathbf{Y}\Big)$ and satisfy*

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\right\|_{\mathbf{Y}}\\&\leq\left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\Big)\right\|_{\mathbf{Y}}\\&+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f\big(x_1\big)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\end{aligned} \tag{3.1}$$

for all $x_j \in \mathbf{X}, j=1\to n$, then f is additive

Proof. Replacing $\big(x_1,...,x_n\big)$ by $\big(0,0...,0\big)$ in (3.1), we get

$$\big(1-|t|\big)\Big\|f\big(0\big)\Big\|\leq 0 \tag{3.2}$$

So $f\big(0\big)=0$.
Next Assume that $f\in A\Big(\mathbf{X},\mathbf{Y}\Big)$satisfies (3.1)
Replacing $\big(x_1,...,x_n\big)$ by $\big(x,0,0...,0\big)$ in (3.1), we get

$$\left\|2f\Big(\frac{x}{2}\Big)-f\big(x\big)\right\|_{\mathbf{Y}}\leq 0$$

and so $2f\Big(\frac{x}{2}\Big)=f\big(x\big)$ for all $x\in\mathbf{X}$.
Thus

$$f\Big(\frac{x}{2}\Big)=\frac{1}{2}f\Big(x\Big) \tag{3.3}$$

for all $x\in\mathbf{X}$ It follows from (3.1) and (3.3) that

$$\begin{aligned}&\left\|f\big(x_1+x_2+\frac{x_3+...+x_n}{2}\big)+f\big(x_1-x_2-\frac{x_3+...+x_n}{2}\big)-2f\big(x_1\big)\right\|_{\mathbf{Y}}\\&=\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\right\|_{\mathbf{Y}}\\&\leq\left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\Big)\right\|_{\mathbf{Y}}\\&+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_2+...+x_k}{2}\Big)-f\big(x_1\big)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\end{aligned} \tag{3.4}$$

for all $x_j\in\mathbf{X}, j=1\to n$ and so

$$\begin{aligned}&\Big(1-\Big|s\Big|\Big)\left\|f\big(x_1+x_2+\frac{x_3+...+x_n}{2}\big)+f\big(x_1-x_2-\frac{x_3+...+x_n}{2}\big)-2f\big(x_1\big)\right\|_{\mathbf{Y}}\\&\leq\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f\big(x_1\big)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\end{aligned} \tag{3.5}$$

Next we letting $u=x_1+x_2+\frac{x_3+...+x_n}{2}, v=x_1-x_2-\frac{x_3+x_4+...+x_k}{2}$ in (3.5), we get

$$\Big(1-\big|s\big|\Big)\Big\|f(u)+f(v)-2f\Big(\frac{u+v}{2}\Big)\Big\|_{\mathbf{Y}}$$
$$\leq \big|t\big|\Big\|f(u)-f\Big(\frac{u+v}{2}\Big)-f\Big(\frac{u-v}{2}\Big)\Big\|_{\mathbf{Y}} \tag{3.6}$$

for all $u, v \in \mathbf{X}$
and so

$$\Big(1-\big|s\big|\Big)\Big\|f(u+v)-f(u)-f(v)\Big\|_{\mathbf{Y}}$$
$$\leq \frac{\big|t\big|}{2}\Big\|f(u+v)+f(u-v)-2f(u)\Big\|_{\mathbf{Y}} \tag{3.7}$$

for all $u, v \in \mathbf{X}$ It follows from (3.5) and (3.7) that

$$\Big(1-\big|s\big|\Big)^2\Big\|f(x_1+x_2+\frac{x_3+...+x_n}{2})-f(x_1)-f(x_2+\frac{x_3+...+x_n}{2})\Big\|_{\mathbf{Y}}$$
$$\leq \frac{\big|t\big|^2}{2}\Big\|f(x_1+x_2+\frac{x_3+...+x_n}{2})-f(x_1)-f(x_2+\frac{x_3+...+x_n}{2})\Big\|_{\mathbf{Y}} \tag{3.8}$$

Since $\sqrt{2}\big|\beta_1\big|+\big|\beta_2\big|<1$

and so

$$f(x_1+x_2+\frac{x_3+...+x_n}{2})=f(x_1)+f(x_2+\frac{x_3+...+x_n}{2})$$

for all $x_j \in \mathbf{X}, j=1\to n$. Thus f is additive. □

Theorem 3.2. *Suppose $\varphi:\mathbf{X}^n\to\mathbb{R}^+\cup\{0\}$ be a function such that*

$$\varphi\Big(\frac{x_1}{2},\frac{x_2}{2},...,\frac{x_n}{2}\Big)\leq\frac{L}{2}\varphi(x_1,x_2,...,x_n) \tag{3.9}$$

for all $x_j\in\mathbf{X}, j=1\to k$ for $L\in\mathbb{R}^+\cup\{0\}$ with $L<1$. And if $f\in F_0\big(\mathbf{X},\mathbf{Y}\big)$ satisfy

$$\Big\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\Big\|_{\mathbf{Y}}$$
$$\leq\Big\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\Big)\Big\|_{\mathbf{Y}}$$
$$+\Big\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\Big\|_{\mathbf{Y}}$$
$$+\varphi(x_1,x_2,...,x_n) \tag{3.10}$$

for all $x_j\in\mathbf{X}, j=1\to n$.
Then there exists a unique mapping $\psi:\in A_0\big(\mathbf{X},\mathbf{Y}\big)$ such that

$$\Big\|f(x)-\psi(x)\Big\|_{\mathbb{Y}}\leq\frac{1}{(1-L)}\varphi(x,0,...,0) \tag{3.11}$$

for all $x\in\mathbf{X}$

Proof. Replacing $(x_1,x_2,...,x_n)$ by $(x,0,...,0)$ in (3.10), we get

$$\Big\|2f\Big(\frac{x}{2}\Big)-f(x)\Big\|_{\mathbb{Y}}\leq\varphi(x,0,...,0) \tag{3.12}$$

for all $x\in\mathbf{X}$.
Consider the set

$$F_0(\mathbf{X},\mathbf{Y})=\mathbb{S}:=\Big\{h:\mathbf{X}\to\mathbf{Y},h(0)=0\Big\}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d\Big(g,h\Big):=inf\Big\{\lambda\in\mathbb{R}:\Big\|g(x)-h(x)\Big\|\leq\lambda\varphi(x,0,...,0),\forall x\in\mathbf{X}\Big\},$$

where, as usual, $inf\phi=+\infty$. It easy to show that $\Big(\mathbb{S},d\Big)$ is complete $(see[17])$ Now we cosider the linear mapping $J:\mathbb{S}\to\mathbb{S}$ such that

$$Jg(x):=2g\Big(\frac{x}{2}\Big)$$

for all $x\in\mathbf{X}$. Let $g,h\in\mathbb{S}$ be given such that $d(g,h)=\epsilon$. Then

$$\Big\|g(x)-h(x)\Big\|\leq\epsilon\varphi(x,0,...,0)$$

for all $x \in \mathbf{X}$.
Hence

$$\Big\|Jg(x) - Jh(x)\Big\| = \Big\|2g\Big(\frac{x}{2}\Big) - 2hf\Big(\frac{x}{2}\Big)\Big\| \leq 2\epsilon\varphi\Big(\frac{x}{2}, 0, ..., 0\Big)$$
$$\leq 2\epsilon\frac{L}{2}\varphi\big(x, 0, ..., 0\big) \leq L\epsilon\varphi\big(x, 0, ..., 0\big)$$

for all $x \in \mathbf{X}$. So $d\big(g, h\big) = \epsilon$ implies that $d\Big(Jg, Jh\Big) \leq L \cdot \epsilon$. This means that

$$d\Big(Jg, Jh\Big) \leq Ld\Big(g, h\Big)$$

for all $g, h \in \mathbb{S}$ It folows from (3.12) that

$$d\Big(f, Jf\Big) \leq 1.$$

By Theorem 2.1, there exists a mapping $\psi : \mathbf{X} \to \mathbf{Y}$ satisfying the flowing:

(1) ψ is a unique fixed point of J, ie.,

$$\psi\big(x\big) = 2\psi\Big(\frac{x}{2}\Big) \tag{3.13}$$

for all $x \in \mathbf{X}$. The mapping ψ is a unique fixed point J in the set

$$\mathbb{M} = \Big\{g \in \mathbb{S} : d\Big(f, g\Big) < \infty\Big\}$$

This implies that ψ is a unique mapping satisfying (3.13) such that there exists a $\lambda \in \Big(0, \infty\Big)$ satisfying

$$\Big\|f\big(x\big) - \psi\big(x\big)\Big\| \leq \lambda\varphi\big(x, 0, ..., 0\big)$$

for all $x \in \mathbf{X}$

(2) $d\Big(J^l f, \psi\Big) \to 0$ as $l \to \infty$. This implies equality

$$\lim_{l\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) = \psi\big(x\big)$$

for all $x \in \mathbb{X}$

(3) $d\Big(f, \psi\Big) \leq \frac{1}{1-L}d\Big(f, Jf\Big)$. which implies

$$\Big\|f\big(x\big) - \psi\big(x\big)\Big\| \leq \frac{1}{1-L}\varphi\big(x, 0, ..., 0\big)$$

for all $x \in X$. It follows (3.9) and (3.10) that

$$\Big\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\Big\|_{\mathbf{Y}}$$
$$=\lim_{n\to\infty}2^n\Big\|2f\big(\frac{x_1+x_2}{2^{n+1}}+\frac{x_3+x_4+...+x_n}{2^{n+2}}\big)-f\big(\frac{x_1}{2^n}-\frac{x_2}{2^n}-\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-2f\big(\frac{x_1}{2^n}\big)\Big\|_{\mathbf{Y}}$$
$$\leq\lim_{n\to\infty}2^n\big|s\big|\Big\|f\big(\frac{x_1+x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-f\big(\frac{x_1-x_2}{2^n}-\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-2f\big(\frac{x_1}{2n}\big)\Big\|_{\mathbf{Y}}$$
$$+\lim_{n\to\infty}2^n\big|t\big|\Big\|f\big(\frac{x_1+x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-f\big(\frac{x_1}{2^n}\big)-f\big(\frac{x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)\Big\|_{\mathbf{Y}}$$
$$+\lim_{n\to\infty}2^n\varphi\big(\frac{x_1}{2^n},\frac{x_2}{2^n},...,\frac{x_n}{2^n}\big)$$
$$=\Big\|s\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+\psi\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2\psi\big(x_1\big)\Big)\Big\|_{\mathbf{Y}}$$
$$+\Big\|t\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-\psi\big(x_1\big)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\Big\|_{\mathbf{Y}} \tag{3.14}$$

for all $x_j\in\mathbb{X}, j=1\to n$. So

$$\Big\|2\psi\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)-\psi\Big(x_1-x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-2\psi\big(x_1\big)\Big\|_{\mathbf{Y}}$$
$$\leq\Big\|s\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+\psi\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2\psi\big(x_1\big)\Big)\Big\|_{\mathbf{Y}}$$
$$+\Big\|t\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-\psi\big(x_1\big)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\Big\|_{\mathbf{Y}}$$

for all $x_j\in\mathbf{X}, j=1\to n$. By *Lemma* 3.1, the mapping $\psi:\mathbf{X}\to\mathbf{Y}$ is additive. Ei

$$\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-\psi\big(x_1\big)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)=0$$

□

Theorem 3.3. *Suppose* $\varphi:\mathbf{X}^n\to\mathbb{R}^+\cup\{0\}$ *be a function such that*

$$\varphi\Big(\frac{x_1}{2},\frac{x_2}{2},...,\frac{x_n}{2}\Big)\leq\frac{L}{2}\varphi\big(x_1,x_2,...,x_n\big) \tag{3.15}$$

for all $x_j \in \mathbf{X}, j = 1 \to k$ *for* $L \in \mathbb{R}^+ \cup \{0\}$ *with* $L < 1$. *And if* $f \in F_0\big(\mathbf{X}, \mathbf{Y}\big)$ *satisfy*

$$\begin{aligned}
&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\right\|_{\mathbf{Y}}\\
&\quad\le\left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f\big(x_1\big)\Big)\right\|_{\mathbf{Y}}\\
&\quad+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f\big(x_1\big)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\
&\quad+\varphi\big(x_1,x_2,...,x_n\big) \qquad (3.16)
\end{aligned}$$

for all $x_j \in \mathbf{X}, j = 1 \to n$.
Then there exists a unique mapping $\psi :\in A_0\big(\mathbf{X}, \mathbf{Y}\big)$ *such that*

$$\left\|f\big(x\big)-\psi\big(x\big)\right\|_{\mathbb{Y}} \le \frac{1}{\big(1-L\big)}\varphi\big(x,0,...,0\big) \qquad (3.17)$$

for all $x \in \mathbf{X}$
Proof. Replacing $\big(x_1, x_2, ..., x_n\big)$ by $\big(x, 0, ..., 0\big)$ in (3.16), we get

$$\left\|2f\Big(\frac{x}{2}\Big)-f\big(x\big)\right\|_{\mathbf{Y}} \le \varphi\big(x,0,...,0\big) \qquad (3.18)$$

for all $x \in \mathbf{X}$.
So

$$\left\|f\big(x\big)-\frac{1}{2}f\big(2x\big)\right\|_{\mathbf{Y}} \le \frac{1}{2}\varphi\big(2x,0,...,0\big) \qquad (3.19)$$

for all $x \in \mathbf{X}$.
Suppose $\big(\mathbb{S} = F_0\big(\mathbf{X}, \mathbf{Y}\big), d\big)$ be the generalized metric space defined in the proof of Theeorem 3.2 Now we cosider the linear mapping $J : \mathbb{S} \to \mathbb{S}$ such that

$$Jg\big(x\big) := \frac{1}{2}g\Big(2x\Big)$$

for all $x \in \mathbf{X}$. That It follows from (3.19)

$$\left\|f\big(x\big)-\frac{1}{2}f\big(2x\big)\right\|_{\mathbf{Y}} \le \frac{1}{2}\varphi\big(2x,0,...,0\big) \le L\varphi\big(x,0,...,0\big)$$

The rest of the proof is similar to proof of Theorem 3.2. □

From proving the theorems we have consequences:

Corollary 3.4. *Suppose $r,\theta \in \mathbb{R}^+ \cup \{0\}$ with $r > 1$. Let $f \in F_0(\mathbf{X},\mathbf{Y})$ and satisfy*

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\right\|_{\mathbf{Y}}\\&\le\left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}\\&+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\&+\theta\Big(\|x_1\|^r+\|x_2\|^r+...+\|x_k\|^r\Big)\end{aligned}\tag{3.20}$$

for all $x_j \in \mathbf{X}$ for all $j = 1 \to k$.
Then there exists a unique mapping $\psi :\in A_0(\mathbf{X},\mathbf{Y})$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \le \frac{2^r\theta}{2^r-2}\left\|x\right\|_{\mathbf{X}}^r \tag{3.21}$$

for all $x \in \mathbf{X}$

Corollary 3.5. *Suppose $r,\theta \in \mathbb{R}^+ \cup \{0\}$ with $r < 1$. Let $f \in F_0(\mathbf{X},\mathbf{Y})$ and satisfy*

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\right\|_{\mathbf{Y}}\\&\le\left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}\\&+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\&+\theta\Big(\|x_1\|^r+\|x_2\|^r+...+\|x_k\|^r\Big)\end{aligned}\tag{3.22}$$

for all $x_j \in \mathbf{X}$ for all $j = 1 \to k$.
Then there exists a unique mapping $\psi :\in A_0(\mathbf{X},\mathbf{Y})$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \le \frac{2^r\theta}{2^r-2}\left\|x\right\|_{\mathbf{X}}^r \tag{3.23}$$

for all $x \in \mathbf{X}$

4. Establish The Solution of The Additive (s,t)-Function Inequalities Using A Direect Method

Now, we first study the solutions of (1.1). Note that for these inequalities, when **X** be a normed space and **Y** is a Banach space.

Theorem 4.1. *Suppose $\varphi : \mathbf{X}^n \to \mathbb{R}^+ \cup \{0\}$ be a mapping such that*

$$\phi(x_1,x_2,...,x_n) := \sum_{j=1}^{\infty} 2^j\varphi\Big(\frac{x_1}{2^j},\frac{x_2}{2^j},...,\frac{x_n}{2^j}\Big) < \infty \tag{4.1}$$

and let $f \in F_0(\mathbf{X},\mathbf{Y})$ satisfies

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\right\|_{\mathbf{Y}}\\&\le\left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}\\&+\left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\&+\varphi(x_1,x_2,...,x_n)\end{aligned}\tag{4.2}$$

for all $x_j \in \mathbf{X}, j = 1 \to n$.
Then there exists a unique mapping $\psi :\in A_0(\mathbf{X},\mathbf{Y})$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \le \varphi(x,0,...,0) \tag{4.3}$$

for all $x \in \mathbb{X}$

Proof. Replacing $(x_1, x_2, ..., x_n)$ by $(x, 0, ..., 0)$ in (4.2), we get

$$\left\|2f(\frac{x}{2}) - f(x)\right\|_{\mathbf{Y}} \leq \varphi(x, x, 0, ..., 0) \tag{4.4}$$

for all $x \in X$
. Hence

$$\begin{aligned}\left\|2^l f\Big(\frac{x}{2^l}\Big) - 2^m f\Big(\frac{x}{2^m}\Big)\right\|_{\mathbf{Y}} &\leq \sum_{j=l}^{m-1} \left\|2^j f\Big(\frac{x}{2^j}\Big) - 2^{j+1} f\Big(\frac{x}{2^{j+1}}\Big)\right\|_{\mathbf{Y}} \\ &\leq \sum_{j=l}^{m-1} 2^j \varphi\Big(\frac{x}{2^{j+1}}, 0, ..., 0\Big)\end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (4.5) that the sequence $\left\{2^n f\Big(\frac{x}{2^n}\Big)\right\}$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{2^n f\Big(\frac{x}{2^n}\Big)\right\}$ coverges. So one can define the mapping $\psi : \mathbb{X} \to \mathbb{Y}$ by

$$\psi(x) := \lim_{n\to\infty} 2^n f\Big(\frac{1}{2^n} x\Big) \tag{4.6}$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.6), we get (4.3)
It follows from (4.1) and (4.2) that

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2} + \frac{x_3+x_4+...+x_k}{4}\Big) + f\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2f(x_1)\right\|_{\mathbf{Y}} \\ &= \lim_{n\to\infty} 2^n \left\|2f(\frac{x_1+x_2}{2^{n+1}} + \frac{x_3+x_4+...+x_n}{2^{n+2}}) - f(\frac{x_1}{2^n} - \frac{x_2}{2^n} - \frac{x_3+x_4+...+x_n}{2^{n+1}}) - 2f(\frac{x_1}{2^n})\right\|_{\mathbf{Y}} \\ &\leq \lim_{n\to\infty} 2^n \Big|s\Big| \left\|f(\frac{x_1+x_2}{2^n} + \frac{x_3+x_4+...+x_n}{2^{n+1}}) - f(\frac{x_1-x_2}{2^n} - \frac{x_3+x_4+...+x_n}{2^{n+1}}) - 2f(\frac{x_1}{2n})\right\|_{\mathbf{Y}} \\ &+ \lim_{n\to\infty} 2^n \Big|t\Big| \left\|f(\frac{x_1+x_2}{2^n} + \frac{x_3+x_4+...+x_n}{2^{n+1}}) - f(\frac{x_1}{2^n}) - f(\frac{x_2}{2^n} + \frac{x_3+x_4+...+x_n}{2^{n+1}})\right\|_{\mathbf{Y}} \\ &+ \lim_{n\to\infty} 2^n \varphi(\frac{x_1}{2^n}, \frac{x_2}{2^n}, ..., \frac{x_n}{2^n}) \\ &= \left\|s\Big(\psi\Big(x_1 + x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) + \psi\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2\psi(x_1)\Big)\right\|_{\mathbf{Y}} \\ &+ \left\|t\Big(\psi\Big(x_1 + x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) - \psi(x_1) - \psi\Big(x_2 + \frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\end{aligned} \tag{4.7}$$

for all $x_j \in \mathbb{X}, j = 1 \to n$. So

$$\begin{aligned}&\left\|2\psi\Big(\frac{x_1+x_2}{2} + \frac{x_3+x_4+...+x_k}{4}\Big) - \psi\Big(x_1 - x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) - 2\psi(x_1)\right\|_{\mathbf{Y}} \\ &\leq \left\|s\Big(\psi\Big(x_1 + x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) + \psi\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2\psi(x_1)\Big)\right\|_{\mathbf{Y}} \\ &+ \left\|t\Big(\psi\Big(x_1 + x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) - \psi(x_1) - \psi\Big(x_2 + \frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\end{aligned}$$

for all $x_j \in \mathbf{X}, j = 1 \to n$. By *Lemma* 3.1, the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ is additive. Ei

$$\psi\Big(x_1 + x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) - \psi(x_1) - \psi\Big(x_2 + \frac{x_3+x_4+...+x_k}{2}\Big) = 0$$

Now, let $\psi' : \mathbb{X} \to \mathbb{Y}$ be another additive mapping satisfying (4.3). Then we have

$$\begin{aligned}\left\|\psi(x) - \psi'(x)\right\| &= \left\|2^q\psi(\frac{x}{2^q}) - 2^q\psi'(\frac{x}{2^q})\right\|_{\mathbb{Y}} \\ &\leq \left\|2^q\psi(\frac{x}{2^q}) - 2^q f(\frac{x}{2^q})\right\| + \left\|2^q\psi'(\frac{x}{2^q}) - 2^q f(\frac{x}{2^q})\right\|_{\mathbb{Y}} \\ &\leq 2^{q+1}\phi\Big(\frac{x}{2^q}, 0, ..., 0\Big)\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in \mathbb{X}$. So we can conclude that $\psi(x) = \psi'(x)$ for all $x \in \mathbb{X}$. This proves the uniqueness of ψ. □

Theorem 4.2. *Suppose* $\varphi : \mathbf{X}^n \to \mathbb{R}^+ \cup \{0\}$ *be a mapping such that*

$$\phi(x_1, x_2, ..., x_n) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi\Big(2^j x_1, 2^j x_2, ..., 2^j x_n\Big) < \infty \tag{4.8}$$

and let $f \in F_0(\mathbf{X}, \mathbf{Y})$ *satisfies*

$$\begin{aligned} &\left\|2f\Big(\frac{x_1+x_2}{2} + \frac{x_3+x_4+...+x_k}{4}\Big) + f\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2f(x_1)\right\|_{\mathbf{Y}} \\ &\le \left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big) + f\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2f(x_1)\Big)\right\|_{\mathbf{Y}} \\ &+ \left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big) - f(x_1) - f\Big(x_2 + \frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}} \\ &+ \varphi(x_1, x_2, ..., x_n) \end{aligned} \tag{4.9}$$

for all $x_j \in \mathbf{X}, j = 1 \to n.$
Then there exists a unique mapping $\psi :\in A_0(\mathbf{X}, \mathbf{Y})$ *such that*

$$\Big\|f(x) - \psi(x)\Big\|_{\mathbf{Y}} \le \varphi(x, 0, ..., 0) \tag{4.10}$$

for all $x \in \mathbb{X}$

Proof. Replacing $(x_1, x_2, ..., x_n)$ by $(x, 0, ..., 0)$ in (4.9), we get

$$\Big\|2f(\frac{x}{2}) - f(x)\Big\|_{\mathbf{Y}} \le \varphi(x, 0, ..., 0) \tag{4.11}$$

for all $x \in X$. So

$$\Big\|f(x) - \frac{1}{2}f\Big(2x\Big)\Big\|_{\mathbf{Y}} \le \frac{1}{2}\varphi(2x, 0, ..., 0) \tag{4.12}$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|\frac{1}{2^l}f\Big(2^l x\Big) - \frac{1}{2^m}f\Big(2^m x\Big)\right\|_{\mathbb{Y}} \\ &\qquad \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\Big(2^j x\Big) - \frac{1}{2^{j+1}}f\Big(2^{j+1}x\Big)\right\|_{\mathbb{Y}} \\ &\qquad \le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi\big(2^j x, 0, ..., 0x\big) \end{aligned} \tag{4.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (4.13) that the sequence $\left\{\frac{1}{2^n}f\Big(2^n x\Big)\right\}$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{\frac{1}{2^n}f\Big(2^n x\Big)\right\}$ coverges. So one can define the mapping $\psi : \mathbb{X} \to \mathbb{Y}$ by

$$\psi\big(x\big) := \lim_{n\to\infty}\frac{1}{2^n}f\Big(2^n x\Big) \tag{4.14}$$

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.12), we get (4.10).
The rest of the proof is similar to the proof of theorem 4.1. □

From proving the theorems we have consequences:

Corollary 4.3. *Suppose* $r, \theta \in \mathbb{R}^+ \cup \{0\}$ *with* $r > 1$. *Let* $f \in F_0(\mathbf{X}, \mathbf{Y})$ *and satisfy*

$$\begin{aligned} &\left\|2f\Big(\frac{x_1+x_2}{2} + \frac{x_3+x_4+...+x_k}{4}\Big) + f\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2f(x_1)\right\|_{\mathbf{Y}} \\ &\le \left\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big) + f\Big(x_1 - x_2 - \frac{x_3+x_4+...+x_k}{2}\Big) - 2f(x_1)\Big)\right\|_{\mathbf{Y}} \\ &+ \left\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big) - f(x_1) - f\Big(x_2 + \frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}} \\ &+ \theta\Big(\big\|x_1\big\|^r + \big\|x_2\big\|^r + ... + \big\|x_k\big\|^r\Big) \end{aligned} \tag{4.15}$$

for all $x_j \in \mathbf{X}$ *for all* $j = 1 \to k$.
Then there exists a unique mapping $\psi :\in A_0(\mathbf{X}, \mathbf{Y})$ *such that*

$$\Big\|f(x) - \psi(x)\Big\|_{\mathbf{Y}} \le \frac{2^r\theta}{2^r - 2}\big\|x\big\|^r_{\mathbf{X}} \tag{4.16}$$

for all $x \in \mathbf{X}$

Corollary 4.4. *Suppose $r, \theta \in \mathbb{R}^+ \cup \{0\}$ with $r < 1$. Let $f \in F_0(\mathbf{X}, \mathbf{Y})$ and satisfy*

$$\begin{aligned}&\Big\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\Big\|_{\mathbf{Y}}\\ &\le \Big\|s\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\Big)\Big\|_{\mathbf{Y}}\\ &+\Big\|t\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\Big\|_{\mathbf{Y}}\\ &+\theta\Big(\|x_1\|^r+\|x_2\|^r+\ldots+\|x_k\|^r\Big) \qquad (4.17)\end{aligned}$$

for all $x_j \in \mathbf{X}$ for all $j = 1 \to k$.
Then there exists a unique mapping $\psi :\in A_0(\mathbf{X}, \mathbf{Y})$ such that

$$\|f(x)-\psi(x)\|_{\mathbf{Y}} \le \frac{2^r\theta}{2^r-2}\|x\|_{\mathbf{X}}^r \qquad (4.18)$$

for all $x \in \mathbf{X}$

CONCLUSION

In this paper, I introduce the general **(s,t)-**function inequality with n variables and then I use two methods of non-zero point and direct direction to prove and show their solutions. This is an unlimited number of variables when we provefunctional inequalities.

Conflicts of Interest

The author declares no conflicts of interest.

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