ON TENSORS AND THE VOLUME ELEMENT OF L_1^n

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Abstract:

In this work, we give the relationship between the alterne tensor $\boldsymbol{det} \in \mathcal{J}^n(\boldsymbol{L_1^n})$ *which is the volume element of* $\boldsymbol{L_1^n}$ *and the symmetric tensor inner product on* $\boldsymbol{L_1^n}$.

1. INTRODUCTION

An important idea underlying general relativity is that space-time which may be described as a curved four-dimensional mathematical structure called a pseudo-Riemannian manifold. A Lorentzian manifold $\boldsymbol{L_1^n}$ is a special case of a pseudo-Riemannian manifold in which the signature of metric is *(1,n-1)*. We need to study on vectors , one-forms and tensors to explain cross product on vectors in $\boldsymbol{L_1^n}$ **.**

Vectors, One-Forms and Tensors

A vector is a quantitiy with a magnitude and a direction. This primitive concept, familiar from undergraduate physics and mathematics, applies equally in general relativity. An example of a vector is $d\vec{x}$, the difference vector between two infinitesimally close points of space-time .Vectors form a linear algebra. This is valid for vectors in a curved four-dimensional space-time as they are vectors in three-dimensional Euclidean space. A one-form is defined as a linear scalar function of a vector. That is, a one-form takes a vector as input and outputs a scalar. For the one-form $\tilde{P}, \tilde{P}(\vec{V})$ is also called the scalar product and denoted $\tilde{P}(\vec{V}) =< \tilde{P}, \vec{V} >$. The one-form is a linear function. We may associate a one-form with a space-time point, resulting in a one-form field $\tilde{P} = \tilde{P}_X$.

$\tilde{P}_X$ is a one-form at point x while $\tilde{P}(\vec{V})$ is a scalar, defined implicitily at point x. One-forms obey their own linear algebra distinct from that of vectors. Vectors and one-forms are linear operators on each other, producing scalars. It is often helpful to consider a vector as being a linear scalar function of a one-form. The vector space of one-forms is called the dual vector (or cotangent) space to distinguish it from the linear space of vectors (tangent space). **[1]**

Definition1.1 If V is a vector space over $\mathbb{R}$ we will denote the k-fold product $V \times \dots \times V$ by V^k. A multilinear function $T: V^k \to \mathbb{R}$ is called a k-tensor on V and the set of all k-tensors, denoted $J^k(V)$, becomes a vector space over $\mathbb{R}$.

Definition1.2 If for $S, T \in J^k(V)$ and $a \in \mathbb{R}$ we define

$$(S+T)(v_1, \dots, v_k) = S(v_1, \dots, v_k) + \mathrm{T}(v_1, \dots, v_k)$$
$$(aS)(v_1, \dots, v_k) = aS(v_1, \dots, v_k)$$

If $S \in J^k(V)$ and $T \in J^l(V)$ we define the tensor product $S \otimes T \in J^{k+l}(V)$ by

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = S(v_1, \dots, v_k)\, \mathrm{T}(v_{k+1}, \dots, v_{k+l})$$

Definition 1.3: If $f: V \to W$ is a linear transformation, a linear transformation $f^*: J^k(W) \to J^k(V)$ is defined by

$$f^*\mathrm{T}(v_1, \dots, v_k) = T\big(f(v_1), \dots, f(v_k)\big)\ for\ T \in J^k(W) and\ v_1, \dots, v_k \in V.$$

There are well known tensors aside from members of dual space V^*. The inner product $<,>\in J^2(\mathbb{R}^n)$ is the first example. The usual inner product on $\mathbb{R}^n$ is symmetric and positive definite 2-tensor. Despite its importance the inner product plays for lesser role than another familiar function, the tensor $\boldsymbol{det} \in \boldsymbol{J^n}(\mathbb{R}^n)$ which is alternating n- tensor. We denote alternating k-tensors on V by $\Lambda^k(V)$ which has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ where n is the dimension of V. If $n = k$ then $\Lambda^n(V)$ has dimension 1.Thus all alternating n-tensors on V are multiples of any non-zero one. Since the determinant is an example of such a member of $\Lambda^n(\mathbb{R}^n)$.

2. PRELIMINARIES

Theorem 2.1 Let $v_1, \dots, v_n$ be a basis for V and let $\omega \in \Lambda^n(V)$. If $\omega_i = \sum_{j=1}^n a_{ij} v_j$ are n vectors in V then $\omega(\omega_1, \dots, \omega_n) = \det(a_{ij})\, \omega(\omega_1, \dots, \omega_n)$. An orientation for V is depend on the sign of det A. If an orientation μ for V has been given, it follows that there is a unique $\omega \in \Lambda^n(V)$ such that $\omega(v_1, \dots, v_n) = 1$ whenever $v_1, \dots, v_n$ is an orthonormal basis such that $[v_1, \dots, v_n] = \mu$. This unique ω is called the volüme element of *V*, determined by the inner product T and orientation μ. Note that det is the volume element of $\mathbb{R}^n$ determined by the usual inner product and usual orientation, and that $|\det(v_1, \dots, v_n)|$ is the volume of the parallelpiped spanned by the line segments from 0 to each $v_1, \dots, v_n$.

If $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ and φ is defined by

$$\varphi(\omega) = det\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ \omega \end{pmatrix}$$ then $\varphi \in \Lambda^1(\mathbb{R}^n)$; therefore there is a unique $z \in \mathbb{R}^n$ such that

$$< \omega, z >= \varphi(\omega) = det\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ \omega \end{pmatrix}$$. This z is denoted $v_1 \times \dots \times v_{n-1}$ and called the cross product of $v_1, \dots, v_{n-1}$.

In the case of two vectors $v_1, v_2 \in \mathbb{R}^3$

$$< \omega, z > = \varphi(\omega) = det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix}$$

$$< \omega, z > = \omega_1 z_1 + \omega_2 z_2 + \omega_3 z_3$$

$$det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix} = \begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix} = \omega_1 \begin{vmatrix} v_{12} & v_{13} \\ v_{22} & v_{23} \end{vmatrix} - \omega_2 \begin{vmatrix} v_{11} & v_{13} \\ v_{21} & v_{23} \end{vmatrix} + \omega_3 \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} \Rightarrow$$

$z_1 = v_{12}v_{23} - v_{22}v_{13}$, $z_2 = v_{21}v_{13} - v_{11}v_{23}$, $z_3 = v_{11}v_{22} - v_{12}v_{21}$ which means

$z = (z_1, z_2, z_3) = v_1 \times v_2$

In the case of one vector $v_1 \in \mathbb{R}^2$

$$< \omega, z > = \omega_1 z_1 + \omega_2 z_2$$

$det\begin{pmatrix} v_1 \\ \omega \end{pmatrix} = \begin{vmatrix} v_{11} & v_{12} \\ \omega_1 & \omega_2 \end{vmatrix} = \omega_2 v_{11} - \omega_1 v_{12} \Rightarrow z_1 = -v_{12}\ and\ z_2 = v_{11}$ which means

$z = (-v_{12}, v_{11})\ is\ v_1 \times$

In the case of three vector $v_1, v_2, v_3 \in \mathbb{R}^4$

$$< \omega, z > = \omega_1 z_1 + \omega_2 z_2 + \omega_3 z_3 + \omega_4 z_4$$

$$det\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \omega \end{pmatrix} = \begin{vmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{vmatrix}$$

$$= -\omega_1|V_{41}| + \omega_2|V_{42}| - \omega_3|V_{43}| + \omega_4|V_{44}|$$

$z_1 = -|V_{41}|$ $z_2 = |V_{42}|$ $z_3 = -|V_{43}|$ $z_4 = |V_{44}|$ which means

$z = (z_1, z_2, z_3, z_4) = v_1 \times v_2 \times v_3$.

By similar way if $\omega \in \Lambda^n(V)$ is a volume element we can define a " cross product" $v_1 \times \dots \times v_{n-1}$ in terms of ω. **[2]**

Definition 2.1 Let L_1^3 be the three dimensional Lorentz-Minkowski space, that is the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $< , >_L$ given by $< , >_L = dx_1^2 + dx_2^2 - dx_3^2$ where (x_1, x_2, x_3) are the canonical coordinates of $\mathbb{R}^3$. Associated to that metric one has the cross product of two vectors $v_1, v_2 \in L_1^3$ given by

$v_1 \times v_2 = (v_{12}v_{23} - v_{22}v_{13}, v_{21}v_{13} - v_{11}v_{23}, v_{21}v_{12} - v_{11}v_{22})$. **[3]**

We can get the same result by using the method in **[2].** Similarly cross product could be defined for one vector $v_1 \in L_1^2$ and three vectors $v_1, v_2, v_3 \in L_1^4$.
In general $n - 1\ vectors\ v_1, \dots, v_{n-1} \in L_1^n$.

3. CROSS PRODUCT OF VECTORS IN $L^n{}_1$

Theorem 3.1 If $v_1, v_2 \in L_1^3$ and φ is defined by $\varphi(\omega) = det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix}$ $and\ \varphi \in \Lambda^1(L_1^3)$ they there is a unique $z \in L_1^3$ such that $< \omega, z >_L = \varphi(\omega) = det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix}$ this z is denoted $v_1 \times_L v_2$ and called the cross product of v_1, v_2.

Proof: Let $v_1, v_2 \in L_1^3$ and φ is defined by $\varphi(\omega) = det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix}$ $where$

$\varphi \in \Lambda^1(L_1^3)\ for\ z \in L_1^3$ we have $< \omega, z >_L = \omega_1 z_1 + \omega_2 z_2 - \omega_3 z_3$ and

$$det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix} = \begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix} = \omega_1 \begin{vmatrix} v_{12} & v_{13} \\ v_{22} & v_{23} \end{vmatrix} - \omega_2 \begin{vmatrix} v_{11} & v_{13} \\ v_{21} & v_{23} \end{vmatrix} + \omega_3 \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}$$

$$< \omega, z >_L = det\begin{pmatrix} v_1 \\ v_2 \\ \omega \end{pmatrix} \Rightarrow \begin{matrix} z_1 = v_{12}v_{23} - v_{22}v_{13} \\ z_2 = v_{21}v_{13} - v_{11}v_{23} \\ z_3 = v_{21}v_{12} - v_{11}v_{12} \end{matrix}$$

It is clear that $z = (z_1, z_2, z_3)$ is the unique element that satisfies the equation and by **[2]** it is the cross product of $v_1 and\ v_2$

Result 1: In the case of one vector $v_1 \in L_1^2$;

$< \omega, z >_L = \omega_1 z_1 - \omega_2 z_2$

$det\begin{pmatrix} v_1 \\ \omega \end{pmatrix} = \begin{vmatrix} v_{11} & v_{12} \\ \omega_1 & \omega_2 \end{vmatrix} = \omega_2 v_{11} - \omega_1 v_{12} \Rightarrow z_1 = -v_{12}\ and\ z_2 = -v_{11}$ which means

$z = (-v_{12}, -v_{11})\ is\ v_1 \times.$

Result 2: In the case of three vectors $v_1, v_2, v_3 \in L_1^4$;

$< \omega, z >_L = \omega_1 z_1 + \omega_2 z_2 + \omega_3 z_3 - \omega_4 z_4$

$$det\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \omega \end{pmatrix} = \begin{vmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{vmatrix}$$

$= -\omega_1|V_{41}| + \omega_2|V_{42}| - \omega_3|V_{43}| + \omega_4|V_{44}|$ where V_{ij} is the minor of the entry ω_j

$z_1 = -|V_{41}|$ $z_2 = |V_{42}|$ $z_3 = -|V_{43}|$ $z_4 = -|V_{44}|$ which means

$z = (z_1, z_2, z_3, z_4) = v_1 \times_L v_2 \times_L v_3.$

Result 3: Similarly in the case of $n-1$ vectors $v_1, \dots, v_{n-1} \in L_1^n$; one can define a cross product $v_1 \times_L \dots \times_L v_{n-1}$ in terms of ω.

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